1.2.extra.1 Let f and g be functions defined on the real numbers with values in the real numbers (i.e. S=T=R). Show that if f composed with g is injective, then f must be injective. Show that if f composed with g is surjective, then g must be surjective. Construct an example of f and g as above such that f composed with g is surjective, but g is not surjective.

Proof of 1.1.2.extra.1:

Let $f \circ g$ be injective (one-to-one). That is, $f \circ g(x) = f \circ g(x') \Leftrightarrow x = x'$

Suppose f is not injective. That is, $\exists t, t' \in R(g), g(x) = t \neq t' = g(x'), x, x' \in \mathbf{R} \ni$ $f(t) = f(t')$

$\Rightarrow f \circ g(x) = f \circ g(x')$ but $g(x) \neq g(x')$, which is a contradiction to $f \circ g$ injective. Q. E. D

$f \circ g$ is surjective. That is, $\forall z \in \mathbf{R}, \exists\ x \in D(f) \bigcap D(g) = D(f \circ g) \ni$ $f \circ g(x) = z.$

Suppose f is not surjective. We know that $\forall x \in D(g), g(x) = t \in D(f)$. Since f is not surjective, $\exists\ z' \in \mathbf{R} \ni f(t) \neq z'\ \forall\ t \in D(f)$. Thus, $\forall x \in D(g)$, $f \circ g(x) = f(g(x)) = f(t) \neq z'$, that is $f \circ g$ is not surjective, which is a contradiction. Therefore, f must be surjective. Q.E.D.

Let $f(x) = \tan(x)$ and let $g(x) = \tan^{-1}(x)$.

Then, $R(g) = \{x \in \mathbf{R} \mid -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\} \neq \mathbf{R}$. Thus, g is not surjective

But $f \circ g(x) = x$, which is surjective.