1.2.8 Let P be the set of polynomials of one real variable. If $p(x)$ is such a polynomial, define $I(p)$ to be the function whose value at x is

$$I(p)(x) = \int_0^x p(t)dt$$

Explain why I is a function from P to P and determine whether it is one-to-one and onto.

Answer to 1.2.8:

Take a general element of $p \in P$. For some n, $p(t) = \alpha_0 + \alpha_1 t + ... + \alpha_n t^n$.

Then $I(p)(x) = \int_0^x (\alpha_0 + \alpha_1 t + ... + \alpha_n t^n)dt = (\alpha_0 t + \frac{\alpha_1 t^2}{2} + ... + \frac{\alpha_n t^{n+1}}{n+1})_0^x$

$= (\alpha_0 x + \frac{\alpha_1 x^2}{2} + ... + \frac{\alpha_n x^{n+1}}{n+1}) - (\alpha_0 (0) + \frac{\alpha_1 (0)}{2} + ... + \frac{\alpha_n 0^n}{n+1})$

$= \alpha_0 x + \frac{\alpha_1 x^2}{2} + ... + \frac{\alpha_n x^{n+1}}{n+1} \equiv p' \in P$

Thus, $I : P \to P$

Let $p = \sum_{k=0}^n \alpha_k t^k$, $p' = \sum_{k=0}^n \alpha_k ' t^k$. Then $p, p' \in P$

$I(p)(x) = \int_0^x (\sum_{k=0}^n \alpha_k t^k)dt = (\sum_{k=0}^n \frac{\alpha_k t^{k+1}}{k+1})_0^x = \sum_{k=0}^n \frac{\alpha_k x^{k+1}}{k+1}$

$I(p)(x) = \int_0^x (\sum_{k=0}^n \alpha_k ' t^k)dt = (\sum_{k=0}^n \frac{\alpha_k ' t^{k+1}}{k+1})_0^x = \sum_{k=0}^n \frac{\alpha_k ' x^{k+1}}{k+1}$

$\sum_{k=0}^n \frac{\alpha_k x^{k+1}}{k+1} = \sum_{k=0}^n \frac{\alpha_k ' x^{k+1}}{k+1} \Leftrightarrow \frac{\alpha_k}{k+1} = \frac{\alpha_k '}{k+1} \Leftrightarrow \alpha_k = \alpha_k ' \ \forall k = 0,...,n$ by definition of polynomial equality. (Two polynomials of degree n are equal iff their coefficients are equal)

$\Rightarrow p = \sum_{k=0}^n \alpha_k t^k = \sum_{k=0}^n \alpha_k ' t^k = p'$ since $\alpha_k = \alpha_k ' \ \forall k = 0,...,n$ and by definition of polynomial equality.

Therefore, I is one-to-one

I is not onto, as there is no $p \in P \ni I(p)(x) = \int_0^x p(t)dt = k \in P \cap \mathbf{R}$

i.e. the range of this function does not include all the real numbers (polynomials of degree zero.)