1.2.4 If A, B and C are sets, prove that $A \setminus (B \bigcup C) = (A \setminus B) \bigcap (A \setminus C)$

Lemma 1.2.4.a: $x \in A \Leftrightarrow x \in A$ and $x \in A$

Proof of lemma 1.2.4.a:

Since $A = A \bigcap A$, we have $A \subset A \bigcap A$ and $A \supset A \bigcap A$. The result follows immediately by definition of subset.

Lemma 1.2.4.b: $x \notin (B \bigcup C) \Leftrightarrow x \notin B$ and $x \notin C$

Proof of lemma 1.2.4.b:

"$\Rightarrow$" Suppose $x \in (B \bigcup C)$. Then, by definition, $x \in B$ or $x \in C$, which is the negation of the statement $x \notin B$ and $x \notin C$. Thus, the contrapositive has been proven, and we have that $x \notin (B \bigcup C) \Rightarrow x \notin B$ and $x \notin C$

"$\Leftarrow$" Suppose $x \in B$ or $x \in C$. Then, by definition, $x \in (B \cup C)$, which is the negation of the statement $x \notin (B \bigcup C)$. Thus, the contrapositive has been proven, and we have that $x \notin B$ and $x \notin C \Rightarrow x \notin (B \bigcup C)$

Proof of exercise 1.2.4:

$x \in A \setminus (B \bigcup C) \Rightarrow x \in A$ and $x \notin (B \bigcup C)$

$\Rightarrow x \in A$ and $x \notin B$ and $x \notin C$ by lemma 1.2.4.b

$\Rightarrow x \in A$ and $x \notin B$ and $x \in A$ and $x \notin C$ by lemma 1.2.4.a

$\Rightarrow x \in (A \setminus B) \bigcap (A \setminus C)$

$\Rightarrow A \setminus (B \bigcup C) \subset (A \setminus B) \bigcap (A \setminus C)$

$x \in (A \setminus B) \bigcap (A \setminus C) \Rightarrow x \in A$ and $x \notin B$ and $x \in A$ and $x \notin C$

$\Rightarrow x \in A$ and $x \notin B$ and $x \notin C$ by lemma 1.2.4.a

$\Rightarrow x \in A$ and $x \notin (B \bigcup C)$ by lemma 1.2.4.b

$\Rightarrow x \in A \setminus (B \bigcup C)$

$\Rightarrow (A \setminus B) \bigcap (A \setminus C) \subset A \setminus (B \bigcup C)$

$\Rightarrow A \setminus (B \bigcup C) = (A \setminus B) \bigcap (A \setminus C)$