1.1.10 Prove that all real numbers x and y satisfy

$$||x| - |y|| \leq |x - y|$$

Lemma 1.1.10.a: $0x = x$

Proof of Lemma 1.1.10.a:

$0x + 0 = 0x$	Additive identity
$0x = (0+0)x$	Additive identity
$(0+0)x = 0x + 0x$	Distributive property
$\Rightarrow 0x + 0 = 0x + 0x$	
$\Rightarrow 0 + 0x = 0x + 0x$	Additive commutativity
$\Rightarrow 0 = 0x$	Proposition 1.1.1.a

Lemma 1.1.10.b: $-x = (-1)x$

Proof of Lemma 1.1.10.b:

$x + (-x) = 0$	Additive inverse
$x + (-1)x = 1x + (-1)x$	Multiplicative identity
$1x + (-1)x = (1 + (-1))x$	Distributive property
$(1 + (-1))x = 0x$	Additive inverse
$0x = 0$	Lemma 1.1.10.a
$\Rightarrow x + (-x) = x + (-1)x$	
$\Rightarrow (-x) + x = (-1)x + x$	Additive commutativity
$\Rightarrow -x = (-1)x$	Proposition 1.1.1.a

Lemma 1.1.10.c: $-(y - x) = x - y$

Proof of Lemma 1.1.10.c:

$(y - x) + (-(y - x)) = 0$	Additive inverse
$0 = 0 + 0 = (y + (-y)) + (x + (-x))$	Additive inverse
$(y + (-y)) + (x + (-x)) = (y + (-x)) + (x + (-y))$	Additive commutativity and additive associativity
$(y + (-x)) + (x + (-y)) = (y - x) + (x - y)$	Definition of "-"
$\Rightarrow (y - x) + (-(y - x)) = (y - x) + (x - y)$	
$\Rightarrow (-(y - x)) + (y - x) = (x - y) + (y - x)$	Additive commutativity
$\Rightarrow -(y - x) = x - y$	Proposition 1.1.1.a

Lemma 1.1.10.d: $|y - x| = |x - y| \ \forall x, y \in \mathbf{R}$

Proof of Lemma 1.1.10.d:

$\lvert y - x\rvert = \lvert -(-(y - x))\rvert$	Lemma 1.1.5.b
$\lvert -(-(y - x))\rvert = \lvert -(x - y)\rvert$	Lemma 1.1.10.c
$\lvert -(x - y)\rvert = \lvert -1(x - y)\rvert$	Lemma 1.1.10.b
$\lvert -1(x - y)\rvert = \lvert -1\rvert \lvert x - y\rvert$	Proposition 1.1.2.b
$\lvert -1\rvert \lvert x - y\rvert = -(-1)\lvert x - y\rvert$	Defn of $\lvert\cdot\rvert$ with $-1 < 0$
$-(-1)\lvert x - y\rvert = 1\lvert x - y\rvert = \lvert x - y\rvert$	Lemma 1.1.5.b and multiplicative identity

Proof of Exercise 1.1.10:

$\lvert x\rvert = \lvert x + 0\rvert = \lvert x + y + (-y)\rvert$	Additive identity

$\| x + y + (-y) \| = \| x - y + y \|$	Additive commutativity
$\| x - y + y \| \leq \| x - y \| + \| y \|$	Triangle inequality
$\Rightarrow \| x \| \leq \| x - y \| + \| y \|$	
$\Rightarrow \| x \| - \| y \| \leq \| x - y \|$	Axiom O4 and defn of "-"
$\| y \| = \| y + 0 \| = \| y + x + (-x) \|$	Additive identity
$\| y + x + (-x) \| = \| y - x + x \|$	Additive commutativity
$\| y - x + x \| \leq \| y - x \| + \| x \|$	Triangle inequality
$\Rightarrow \| y \| \leq \| y - x \| + \| x \|$	
$\Rightarrow \| y \| - \| x \| \leq \| y - x \| = \| x - y \|$	Axiom O4 ,defn of "-", and lemma 1.1.10.d
$\| y \| - \| x \| = \| -(\| x \| - \| y \|) \|$	Lemma 1.1.10.c

$$\|\, x \,|-|\, y \,\| = \langle \begin{matrix} |\, x \,|-|\, y \,|\ when\ |\, x \,|-|\, y \,| \geq 0 \\ -(|\, x \,|-|\, y \,|)\ when\ |\, x \,|-|\, y \,| < 0 \end{matrix}$$, both of which are $\leq |\, x - y \,|$

Therefore, $\|\, x \,| - |\, y \,\| \leq |\, x - y \,|$ Q.E.D.