4. Show that the function $f(x) = x\sqrt{x}$ is continuous on $[0,\infty)$ but not uniformly continuous.

Answer to question 4:

Consider the function $f_1(x) = x$. Take an arbitrary sequence $\{x_n\} \subset [0,\infty)$ $\ni x_n \to c \in [0,\infty)$ as $n \to \infty$. Then $f(x_n) = x_n \to c = f(c)$ as $n \to \infty$. Therefore, f_1 is continuous at c. Since c is arbitrary, it follows that f_1 is continuous on $[0,\infty)$.

Consider the function $f_2(x) = \sqrt{x}$. Suppose f_2 is not continuous at $x = 0$. Then $\exists \varepsilon_0 > 0 \ni \forall \delta > 0, \exists\, x \ni |x| < \delta$ and $\left|\sqrt{x}\right| > \varepsilon_0$.

Let $\delta = {\varepsilon_0}^2$. Then $x < {\varepsilon_0}^2 \Rightarrow \sqrt{x} < \varepsilon_0$, which is a contradiction $\rightarrow\leftarrow$. Therefore, f_2 is continuous at $x = 0$.

Consider the interval $(0,\infty)$. Take $c \in (0,\infty)$ arbitrary. Take $\varepsilon > 0$ arbitrary.

Let $|x - c| \le \delta \le \varepsilon\sqrt{c}$

$\Rightarrow \left|\sqrt{x} - \sqrt{c}\right| = \dfrac{|x-c|}{\sqrt{x}+\sqrt{c}} \le \dfrac{|x-c|}{\sqrt{c}} \le \dfrac{\delta}{\sqrt{c}} = \dfrac{\varepsilon\sqrt{c}}{\sqrt{c}} = \varepsilon$. Therefore, f_2 is continuous at c. Since c was arbitrary, it follows that f_2 is continuous on $(0,\infty)$.

By theorem 3.1.1.c, we have that $f(x) = f_1 f_2(x)$ is the product of continuous functions on $[0,\infty)$ and is therefore continuous on $[0,\infty)$.

Let $\{x_n\}$ and $\{y_n\}$ be such that $x_n = n^2$ and $y_n = n^2 + \dfrac{1}{n}$.

Then as $n \to \infty$, $|x_n - y_n| = \left|n^2 - n^2 - \dfrac{1}{n}\right| = \left|\dfrac{1}{n}\right| \to 0$. That is, $\forall \delta > 0\ \exists\, N(\varepsilon) \in \mathbf{N}$ $\ni \forall n \ge N, |x_n - y_n| < \delta$.

But $|f(x_n) - f(c_n)| = \left|n^3 - \left(n^2 + \dfrac{1}{n}\right)\sqrt{\left(n^2 + \dfrac{1}{n}\right)}\right| = \left|n^3 - n^2\sqrt{n^2 + \dfrac{1}{n}} - \dfrac{1}{n}\sqrt{n^2 + \dfrac{1}{n}}\right|$

$$= \left|n^3 - \sqrt{n^6 + n^3} - \sqrt{1 + \frac{1}{n^3}}\right| = \left|\frac{n^6 - n^6 - n^3}{n^3 + \sqrt{n^6 + n^3}} - \sqrt{1 + \frac{1}{n^3}}\right|$$

$$= \left|\frac{n^3}{n^3 + \sqrt{n^6 + n^3}} + \sqrt{1 + \frac{1}{n^3}}\right| = \left|\frac{1}{1 + \sqrt{1 + 1/n^3}} + \sqrt{1 + \frac{1}{n^3}}\right|$$

$$= \left| \frac{2 + 1/n^3 + \sqrt{1 + 1/n^3}}{1 + \sqrt{1 + 1/n^3}} \right| \to \frac{3}{2} \text{ as } n \to \infty.$$ That is, $\exists \varepsilon > 0 \ni \forall N, \exists\, n \geq N$ satisfying $|f(x_n) - f(y_n)| \geq \varepsilon$. Therefore, f is not uniformly continuous on $[0, \infty)$.