

4. Let $\{x_n\}$ be a sequence defined by the recurrence $x_1 = 1$, $x_{n+1} = x_n + \frac{\sin(x_n)}{2^n}$. Use the definition to show that this sequence is Cauchy.

Lemma 4:

The sequence $\frac{1}{2^{n-1}} \rightarrow 0$ as $n \rightarrow \infty$.

Proof of lemma 4:

Let $\varepsilon > 0$ arbitrary. Want to find some $N(\varepsilon) \in \mathbf{N} \ni \forall n \geq N$, $\left| \frac{1}{2^{n-1}} \right| < \varepsilon$.

$$\left| \frac{1}{2^{n-1}} \right| = \frac{1}{2^{n-1}} < \varepsilon \quad \forall n \geq N \Leftrightarrow 2^{1-n} < \varepsilon \quad \forall n \geq N \Leftrightarrow (1-n) \ln 2 < \varepsilon \quad \forall n \geq N$$

$$\Leftrightarrow \ln 2 - n \ln 2 < \varepsilon \quad \forall n \geq N \Leftrightarrow \ln 2 - \varepsilon < n \ln 2 \quad \forall n \geq N$$

$$\Leftrightarrow n > \frac{\ln 2 - \varepsilon}{\ln 2} \quad \forall n \geq N. \text{ Thus, we can pick } N \geq \frac{\ln 2 - \varepsilon}{\ln 2}. \text{ Then, } \forall n \geq N, \text{ we}$$

have: $\left| \frac{1}{2^{n-1}} \right| < \varepsilon$. Q.E.D.

Proof of question 4:

$$|x_{n+1} - x_n| = \left| x_n + \frac{\sin(x_n)}{2^n} - x_n \right| = \left| \frac{\sin(x_n)}{2^n} \right| \leq \left| \frac{1}{2^n} \right| = \frac{1}{2^n}. \text{ Take } m > n:$$

$$\Rightarrow |x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \dots + x_{n+1} - x_n|$$

$$\leq |x_m - x_{m-1}| + \dots + |x_{n+1} - x_n| \leq \frac{1}{2^{m-1}} + \dots + \frac{1}{2^n} = \frac{1}{2^n} \left(1 + \dots + \frac{1}{2^{m-n-1}} \right)$$

$$\leq \frac{1}{2^n} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots \right) = \frac{1}{2^n} \left(\frac{1}{1 - \frac{1}{2}} \right) = \frac{2}{2^n} = \frac{1}{2^{n-1}}. \text{ (The convergence of geometric}$$

series is assumed. For a proof, see Problem Set 3, question 2.4.7)

Thus, $\forall m > n \geq N$, we have:

$$|x_m - x_n| \leq \frac{1}{2^{n-1}} < \varepsilon. \text{ That is, } \{x_n\} \text{ is a Cauchy sequence. Q.E.D.}$$