2 For each $k \in \mathbf{N}$, denote by

$$\mathbf{FN}_k = \{A \mid A \subset \mathbf{N},\ A \text{ has exactly } k \text{ elements}\}$$
$$\mathbf{FN} = \{A \mid A \subset \mathbf{N},\ A \text{ is finite and nonempty}\}$$
$$\mathbf{PN} = \{A \mid A \subset \mathbf{N}\}$$
$$\mathbf{IN} = \{A \mid A \subset \mathbf{N},\ A \text{ is infinite and } \mathbf{N} \setminus A \text{ is infinite}\}$$

2.a Show that $\mathbf{FN} = \bigcup_{k\geq 1} \mathbf{FN}_k$

Proof of question 2.a:

Take $A \in \mathbf{FN}$. Then $A \subset N$, A is finite and A is nonempty.

$\Rightarrow A \subset N,\ \#A = k \geq 1 \Rightarrow A \subset NA$ and $\#A = 1$ or $\#A = 2$ or $\#A = 3$ or…

$\Rightarrow A \in \bigcup_{k\geq 1} \mathbf{FN}_k \Rightarrow \mathbf{FN} \subset \bigcup_{k\geq 1} \mathbf{FN}_k$

Take $A \in \bigcup_{k\geq 1} \mathbf{FN}_k$. Then $A \subset N$ and $\#A = 1$ or $\#A = 2$ or $\#A = 3$ or…

$\Rightarrow A \subset N$ and A is finite

$\Rightarrow A \in \mathbf{FN} \Rightarrow \bigcup_{k\geq 1} \mathbf{FN}_k \subset \mathbf{FN}$

$\Rightarrow \mathbf{FN} = \bigcup_{k\geq 1} \mathbf{FN}_k$

2.b Show that each set $\mathbf{FN}_k$ is countable, by explicitly showing a bijection between $\mathbf{FN}_k$ and the Cartesian product of k copies of $\mathbf{N}$, $\mathbf{N} \times \ldots \times \mathbf{N}$.

Lemma 2.b.a:

$\mathbf{N} \times \ldots \times \mathbf{N}$ (*n times*) is countable

Proof of lemma 2.b.a: (by induction)

$(n = 1)$: $\mathbf{N}$ is countable. This is trivially true.

Assume $(n = n)$ is true: $\mathbf{N} \times \ldots \times \mathbf{N}$ (*n times*) is countable.

$(n = n+1)$: $\mathbf{N} \times \ldots \times \mathbf{N} \times \mathbf{N}$ $(n+1$ *times*$) = (\mathbf{N} \times \ldots \times \mathbf{N}) \times \mathbf{N}$. By $(n = 1)$ and $(n = n)$, we have that $(\mathbf{N} \times \ldots \times \mathbf{N}) \times \mathbf{N}$ is the Cartesian product of two countable sets. By proposition 1.3.4, it is countable. Thus, $(n = n+1)$ is true. Q.E.D

Proof of question 2.b:

Fix k. Then $\mathbf{FN}_k = \{A \mid A \subset \mathbf{N}, \#A = k\}$.

Since each $A \in \mathbf{FN}_k$ has exactly k elements, we can arrange them as:

A=$\{a_1, \ldots, a_k\}$.

Define $f : \mathbf{FN}_k \to \mathbf{N} \times \ldots \times \mathbf{N}$ (k *times*) by $f(A) = (a_1, a_2, \ldots, a_k)$.

Take $f(A) = f(A') \Rightarrow (a_1, \ldots, a_k) = (a_1', \ldots, a_k')$

$\Rightarrow a_1 = a_1',\ a_2 = a_2',\ \ldots,\ a_k = a_k' \Rightarrow A = A'$. Thus, *f* is injective.

f is not surjective since we can find some $n, \in \mathbf{N} \times \ldots \times \mathbf{N}$ (k *times*) such that $n_j = n_{j'}$ for $j \neq j'$. Such an element would not be in the image of *f*. But if we restrict the codomain of *f* by: $f : \mathbf{FN}_k \to R(f)$, we have a bijection. Since $R(f)$ is an infinite subset of a countable set, it is countable by proposition 1.3.2.

Defined this way, f is a bijection. Thus, for each k, $\mathbf{FN}_k$ is countable. Q.E.D.

2.c Conclude that $\mathbf{FN}$ is countable.

Answer to question 2.c:

From part b, we have established that each set $\mathbf{FN}_k$ is countable. From part a, we have established that $\mathbf{FN} = \bigcup_{k \geq 1} \mathbf{FN}_k$, i.e. $\mathbf{FN}$ is the countable union of countable sets. Thus, $\mathbf{FN}$ is countable.

2.d Prove by contradiction that $\mathbf{IN}$ is not countable. (Hint: Use the fact that $\mathbf{PN}$ is not countable, and part (c).)

Proof of question 2.d:

We know that $\mathbf{IN} = \mathbf{PN} \cup \mathbf{FN} \cup (\mathbf{IN} \setminus \mathbf{PN} \cup \mathbf{FN})$. For the same reasons as given in a, b, and c, we have that $\mathbf{IN} \setminus (\mathbf{PN} \cup \mathbf{FN})$ is countable.

Assume $\mathbf{IN}$ is countable. Then $\mathbf{PN}$ is countable, which is a contradiction $\rightarrow\leftarrow$. Therefore, $\mathbf{IN}$ is uncountable. Q.E.D.