Quantative Methods
Spring 2005

1 Part I Q1

The MGF of Y_n is given by $M(t) = (1 - \frac{\pi}{n} + \frac{\pi}{n}e^t)^n = (1 + (e^t - 1)\frac{\pi}{n})^n$. As $n \to \infty$, the limit of the MGF is

$$\exp(\pi(e^t - 1)),$$

which is the MGF of Poisson with mean π. Note that the MGF is unique.■

2 Part I Q2

(a)

It is clear that $\overline{Z} = \frac{\overline{X} - \mu}{\sigma}$. So we have $X_i - \overline{X} = \sigma(Z_i - \overline{Z})$. Plugging this into Y_n gives

$$\begin{aligned} Y_n &= \frac{\sigma^4 \frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^4}{\sigma^4 \left(\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^2\right)^2} - 3 \\ &= \frac{\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^4 - 3\left(\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^2\right)^2}{\left(\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^2\right)^2}.■ \end{aligned}$$

(b)

By the hint, $\left(\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^2\right)^2 = \left(\frac{1}{n}\sum_{i=1}^n Z_i^2 - \left(\frac{1}{n}\sum_{i=1}^n Z_i\right)^2\right)^2$. Since $Z_i \overset{iid}{\sim} N(0,1)$, $Z_i^2 \overset{iid}{\sim} \chi(1)$.Note that $E(Z_i) = 0, E(Z_i^2) = 1$ and $Var(Z_i) = 1 < \infty$, $Var(Z_i^2) = 2 < \infty$. By the WLLN, $\frac{1}{n}\sum_{i=1}^n Z_i^2 = 1 + o_p(1)$ and $\frac{1}{n}\sum_{i=1}^n Z_i = 0 + o_p(1)$. By Slutsky Theorem,

$$\begin{aligned} \left(\frac{1}{n}\sum_{i=1}^n Z_i^2 - \left(\frac{1}{n}\sum_{i=1}^n Z_i\right)^2\right)^2 &= \left(1 + o_p(1) - \left(0^2 + o_p(1)\right)\right)^2 \\ &= \left(1 + o_p(1)\right)^2 \\ &= 1^2 + o_p(1).■ \end{aligned}$$

(c)

Define $W_i = \begin{bmatrix} Z_i^4 \\ Z_i^3 \\ Z_i^2 \\ Z_i \end{bmatrix}$. W_i is i.i.d. By the hint, $E[W_i] = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Also,

$$\begin{aligned} Var(W_i) &= E[W_i W_i'] - E[W_i]E[W_{i'}'] \\ &= E\begin{bmatrix} Z_i^8 & Z_i^7 & Z_i^6 & Z_i^5 \\ Z_i^7 & Z_i^6 & Z_i^5 & Z_i^4 \\ Z_i^6 & Z_i^5 & Z_i^4 & Z_i^3 \\ Z_i^5 & Z_i^4 & Z_i^3 & Z_i^2 \end{bmatrix} - \begin{bmatrix} 9 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 105 & 0 & 15 & 0 \\ 0 & 15 & 0 & 3 \\ 15 & 0 & 3 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 9 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 96 & 0 & 12 & 0 \\ 0 & 15 & 0 & 3 \\ 12 & 0 & 2 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}. \end{aligned}$$

We want to know the limiting distribution of $\sqrt{n}(\frac{1}{n}\sum W_i - E[W_i])$. By CLT,

$$\sqrt{n}(\frac{1}{n}\sum W_i - E[W_i]) \xrightarrow{d} N(0, Var(W_i)) = N\left(0, \begin{bmatrix} 96 & 0 & 12 & 0 \\ 0 & 15 & 0 & 3 \\ 12 & 0 & 2 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}\right). \blacksquare$$

(d)

CAUTION: we want to know the limiting distribution of $\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^4 - 3\left(\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^2\right)^2\right]$, not the one of $\frac{1}{\sqrt{n}}\left[\sum_{i=1}^n (Z_i - \overline{Z})^4 - 3\left(\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^2\right)^2\right]$ stated in the question.

Define $\overline{W} = \frac{1}{n}\sum_i W_i = \begin{bmatrix} \frac{1}{n}\sum_i Z_i^4 \\ \frac{1}{n}\sum_i Z_i^3 \\ \frac{1}{n}\sum_i Z_i^2 \\ \frac{1}{n}\sum_i Z_i \end{bmatrix} \equiv \begin{bmatrix} A_n \\ B_n \\ C_n \\ D_n \end{bmatrix}$. By the hint,

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^4 - 3\left(\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^2\right)^2 &= A_n - 4D_nB_n + 12D_n^2C_n - 3C_n^2 - 6D_n^4 \\ &\equiv u(\overline{W}). \end{aligned}$$

Note that $u(E[W_i]) = 3 - 3 \cdot 1^2 = 0$. We have

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^4 - 3\left(\frac{1}{n}\sum_{i=1}^n (Z_i - \overline{Z})^2\right)^2\right] = \sqrt{n}\left[u(\overline{W}) - u(E[W_i])\right].$$

The first derivative of u evaluated at $\overline{W} = E[W_i]$ is

$$\begin{aligned} \left.\frac{\partial u}{\partial \overline{W}'}\right|_{\overline{W}=E[W_i]} &= \left.\begin{bmatrix} 1 & -4D_n & 12D_n^2 - 6C_n & -4B_n + 24D_nC_n - 24D_n^3 \end{bmatrix}\right|_{\overline{W}=E[W_i]} \\ &= \begin{bmatrix} 1 & 0 & -6 & 0 \end{bmatrix} \equiv G. \end{aligned}$$

Recall $\sqrt{n}(\overline{W}-E[W_i]) \xrightarrow{d} N(0, Var(W_i))$.By the Delta method, we have

$$\sqrt{n}(u(\overline{W})-u(E[W_i])) \xrightarrow{d} N(0, GVar(W_i)G').$$

So we have

$$\begin{aligned} \omega &= GVar(W_i)G' \\ &= \begin{bmatrix} 1 & 0 & -6 & 0 \end{bmatrix} \begin{bmatrix} 96 & 0 & 12 & 0 \\ 0 & 15 & 0 & 3 \\ 12 & 0 & 2 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -6 & 0 \end{bmatrix}' \\ &= 24. \blacksquare \end{aligned}$$

(e)

$\sqrt{n}Y_n = \frac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n (Z_i-\overline{Z})^4 - 3\left(\frac{1}{n}\sum_{i=1}^n (Z_i-\overline{Z})^2\right)^2\right)}{\left(\frac{1}{n}\sum_{i=1}^n (Z_i-\overline{Z})^2\right)^2}$. Since $\left(\frac{1}{n}\sum_{i=1}^n (Z_i-\overline{Z})^2\right)^2 = 1 + o_p(1)$, Slutsky Theorem gives $\frac{1}{\left(\frac{1}{n}\sum_{i=1}^n (Z_i-\overline{Z})^2\right)^2} = 1^{-1} + o_p(1)$. Also, note that $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n (Z_i-\overline{Z})^4 - 3\left(\frac{1}{n}\sum_{i=1}^n (Z_i-\overline{Z})^2\right)^2\right) \xrightarrow{d} N(0, 24)$. By Slutsky, we have

$$\sqrt{n}Y_n \xrightarrow{d} N(0, 24). \blacksquare$$