3. What is the MLE of $\Pr[x_{10} = 1\ or\ 2]$

$\arg\max_{\theta>0} f(x_1, x_2, \cdots, x_{10}) = \arg\max_{\theta>0} \log f(x_1, x_2, \cdots, x_{10})$

$\log f(x_1, x_2, \cdots, x_{10}) = \log \Pi_{i=1}^{10} \frac{\theta^{x_i} e^{-\theta}}{x_i!} = \log\theta * \sum_{i=1}^{10} x_i - 10\theta - \sum_{i=1}^{10} \log x_i!$

$F.O.C. \qquad \frac{\sum_{i=1}^{10} x_i}{\theta} - 10 = 0 \Rightarrow \hat{\theta}_{mle} = \frac{\sum_{i=1}^{10} x_i}{10} = \bar{x}$

$\Pr[x_{10} = 1\ or\ 2] = \theta e^{-\theta} + \frac{\theta^2 e^{-\theta}}{2}$

By the invariant principle of MLE, the MLE of $\Pr[x_{10} = 1\ or\ 2] = \bar{x}e^{-\bar{x}} + \frac{\bar{x}^2 e^{-\bar{x}}}{2}$.

4.

$\bar{X} = n^{-1} \sum_{i=1}^{10} X_i$

Since $X_1, X_2, \cdots, X_n$ is an iid sample from $Binomial(1, \pi)$, we know that the variance of $\bar{X} = n^{-1} var(X_i) = \frac{\pi(1-\pi)}{n}$.

$\log\mathcal{L} = \log\ f(x_1, x_2, \cdots, x_n)$
$= \log\ \Pi_{i=1}^{n} \pi^{x_i}(1-\pi)^{1-x_i}$
$= \log\pi * \sum_{i=1}^{n} x_i + \log(1-\pi) * (n - \sum_{i=1}^{n} x_i)$

$\frac{\partial \log\mathcal{L}}{\partial\pi} = \frac{\sum_{i=1}^{n} x_i}{\pi} - \frac{n - \sum_{i=1}^{n} x_i}{1-\pi}$
$\frac{\partial^2 \log\mathcal{L}}{\partial\pi^2} = -\frac{\sum_{i=1}^{n} x_i}{\pi^2} - \frac{n - \sum_{i=1}^{n} x_i}{(1-\pi)^2}$

The Fisher Information $I(\pi) = -E[\frac{\partial^2 \log\mathcal{L}}{\partial\pi^2}] = \frac{n\pi}{\pi^2} + \frac{n - n\pi}{(1-\pi)^2} = \frac{n}{\pi(1-\pi)}$

We know that for all unbiased estimator of π, say $E[h(x)] = \pi$, $var[h(x)] \geq I(\pi)^{-1} = \frac{\pi(1-\pi)}{n}$, which is called the Cramer-Rao lower bound.

Hence, $\bar{X} = n^{-1} \sum_{i=1}^{10} X_i$ has the minimum variance among all the unbiased estimators of π.