Quantitative Methods Comprehensive Examination

Comp 2004 Fall, Part I

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1 Question 3

Let $X_1, ..., X_n$ be i.i.d $N(\theta, \sigma^2)$. Suppose that σ^2 is **known**. Show that $\overline{X} = n^{-1} \sum\limits_{i=1}^{n} X_i$ has the minimal variance among all unbiased estimators. Justify your answer by explicitly involing some theorem(s).

1.1 Solution

First we prove that $\overline{X} = n^{-1} \sum\limits_{i=1}^{n} X_i$ is unbiased with $E\left[\overline{X}\right] = \theta$ and $Var(\overline{X}) = \frac{\sigma^2}{n}$:

$E\left[\overline{X}\right] = E\left[n^{-1} \sum\limits_{i=1}^{n} X_i\right] = \frac{1}{n} E\left[\sum\limits_{i=1}^{n} X_i\right] = \frac{1}{n}\left[n\theta\right] = \theta$, Hence $\overline{X}$ is unbiased.

$Var(\overline{X}) = Var\left(n^{-1} \sum\limits_{i=1}^{n} X_i\right) = \frac{1}{n^2} Var\left(\sum\limits_{i=1}^{n} X_i\right) = \frac{1}{n^2}\left(n\sigma^2\right) = \frac{\sigma^2}{n}$

By the Cramer Rao Lower Bound Theorem: If $\overline{X}$ is an unbiased estimator of θ, then $\overline{X}$ has the minimal variance among all unbiased estimators if: $I_n^{-1}(\theta) = Var(\overline{X}\ )$.

Obtaining the Fisher Information for one variable:

$f(x_i, \theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(x_i-\theta)^2}{2\sigma^2}\right]$

$\log f(x_i, \theta) = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{(x_i-\theta)^2}{2\sigma^2}$

$\frac{\delta \log f(x_i,\theta)}{\delta\theta} = \frac{(x_i-\theta)}{\sigma^2}$

$\frac{\delta^2 \log f(x_i,\theta)}{\delta\theta^2} = \frac{-1}{\sigma^2}$

$I_1(\theta) = -E\left[\frac{-1}{\sigma^2}\right] = \frac{1}{\sigma^2}$

And we know that the Fisher Information in a Random Sample of size n equals n times the Fisher Information of one observation, hence:

$I_n(\theta) = n \cdot I_1(\theta) = \frac{n}{\sigma^2}$

And from here we showed that

$Var(\overline{X}) = I_n^{-1}(\theta) = \frac{\sigma^2}{n}$

2 Question 4

Suppose that the moment generating function $E\left[\exp\left(t \cdot X\right)\right]$ of a random variable X is equal to $e^{\mu(e^t-1)}$ for some $\mu > 0$. Show that $Var(X) = E\left[X\right]$.

2.1 Solution

From the moment generating function, we know we are dealing with a Poisson distribution, and we also know that for this distribution:

$Var(X) = E\left[X\right] = \mu$

To show this, we can take the first derivative of the mgf w.r.t. t and evaluate it when $t = 0$, this will give us $E\left[X\right]$. Then, we take the second derivative of the mgf again wrt t and evaluate it when $t = 0$, this will give us $E\left[X^2\right]$. Finally, we know that $Var(X) = E\left[X^2\right] - \left(E\left[X\right]\right)^2$:

$\frac{\delta M(t \cdot X)}{\delta t}|_{t=0} = e^{\mu e^t - \mu} \cdot e^t \mu \mid_{t=0} : e^{\mu e^0 - \mu} \cdot e^0 \mu = e^0 \cdot e^0 \mu = \mu = E\left[X\right]$

$\frac{\delta^2 M(t \cdot X)}{\delta t^2}|_{t=0} = e^t \mu (e^{\mu e^t - \mu} \cdot e^t \mu) + \mu e^{\mu e^t - \mu} \cdot e^t|_{t=0} : e^0 \mu (e^{\mu - \mu} \cdot e^0 \mu) + \mu e^{\mu - \mu} \cdot e^0 = \mu^2 + \mu = E\left[X^2\right]$

$Var(X) = E\left[X^2\right] - \left(E\left[X\right]\right)^2 = \mu^2 + \mu - \mu^2 = \mu$

$\Rightarrow Var(X) = E\left[X\right] = \mu$