Quantitative Methods Comprehensive Examination

Comp 2003 Spring, Part I

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Jan. 31st 2006

1 Question 1

Suppose that $X \sim N(\theta_1, \theta_2)$. Let $\theta = (\theta_1, \theta_2)'$. Compute the Fisher Imformation for θ.

1.1 Method

There are two methods for calculating multi-parameter Fisher Information: (i) by calculating score (ii) by taking partial derivatives twice. Even though (i) is analytically important, it takes some time. Thus, I use method (ii). Note that (i) and (ii) are equivalent.

$$I(\theta) = E[s(X;\theta)s(X;\theta)'] = -E[\frac{\partial^2 \log f(x;\theta)}{\partial\theta\partial\theta'}] = -E\begin{bmatrix} \frac{\partial^2 \log f(X;\theta_1,\theta_2)}{\partial(\theta_1)^2} & \cdots & \frac{\partial^2 \log f(x;\theta_1,\theta_2)}{\partial\theta_1\partial\theta k} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \log f(x;\theta_1,\theta_2)}{\partial\theta_k\partial\theta 1} & \cdots & \frac{\partial^2 \log f(X;\theta_1,\theta_2)}{\partial(\theta_1)^2} \end{bmatrix}$$

1.2 Solution

Since X is normal

$$f(x;\mu,\sigma^2) = f(x;\theta_1,\theta_2) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp[-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}].$$

Replacing (μ,σ^2) by (θ_1,θ_2),

$$f(x;\theta_1,\theta_2) = \frac{1}{\sqrt{2\pi\theta_2}}\exp[-\frac{1}{2}\frac{(x-\theta_1)^2}{\theta_2}].$$

Note that $E(X) = \theta_1$ and $E[(X-\theta_1)^2] = \theta_2$.
Then,

$$\begin{aligned} \log f(x;\theta_1,\theta_2) &= \log\{\tfrac{1}{\sqrt{2\pi\theta_2}}\exp[-\tfrac{1}{2}\tfrac{(x-\theta_1)^2}{\theta_2}]\} \\ &= \log 1 - \log\{\sqrt{2\pi\theta_2}\} + \log\{\exp[-\tfrac{1}{2}\tfrac{(x-\theta_1)^2}{\theta_2}]\} \\ &= -\tfrac{1}{2}\log(2\pi\theta_2) - \tfrac{1}{2}\tfrac{(x-\theta_1)^2}{\theta_2} \\ &= --\tfrac{1}{2}\log(2\pi) - \tfrac{1}{2}\log\theta_2 - \tfrac{1}{2}\tfrac{(x-\theta_1)^2}{\theta_2} \end{aligned}$$

Calculating second derivatives,

$$\begin{aligned} \tfrac{\partial \log f(x;\theta_1,\theta_2)}{\partial\theta_1} &= \tfrac{(x-\theta_1)}{\theta_2} \\ \tfrac{\partial^2 \log f(x;\theta_1,\theta_2)}{\partial(\theta_1)^2} &= -\tfrac{1}{\theta_2} \\ \tfrac{\partial^2 \log f(x;\theta_1,\theta_2)}{\partial\theta_1\partial\theta_2} &= -\tfrac{(x-\theta_1)}{(\theta_2)^2} \\ \tfrac{\partial \log f(x;\theta_1,\theta_2)}{\partial\theta_2} &= -\tfrac{1}{2}\tfrac{1}{\theta_2} + \tfrac{1}{2}\tfrac{(x-\theta_1)^2}{(\theta_2)^2} \\ \tfrac{\partial^2 \log f(x;\theta_1,\theta_2)}{\partial(\theta_2)^2} &= \tfrac{1}{2}\tfrac{1}{(\theta_2)^2} - \tfrac{(x-\theta_1)^2}{(\theta_2)^3} \end{aligned}$$

Then the Fisher information matrix is

$$\begin{aligned}
I(\theta) &= -E\begin{bmatrix} \frac{\partial^2 \log f(X;\theta_1,\theta_2)}{\partial(\theta_1)^2} & \frac{\partial^2 \log f(X;\theta_1,\theta_2)}{\partial\theta_1\partial\theta_2} \\ \frac{\partial^2 \log f(X;\theta_1,\theta_2)}{\partial\theta_2\partial\theta_1} & \frac{\partial^2 \log f(X;\theta_1,\theta_2)}{\partial(\theta_2)^2} \end{bmatrix} \\
&= -E\begin{bmatrix} -\frac{1}{\theta_2} & -\frac{(X-\theta_1)}{(\theta_2)^2} \\ -\frac{(X-\theta_1)}{(\theta_2)^2} & \frac{1}{2}\frac{1}{(\theta_2)^2} - \frac{(X-\theta_1)^2}{(\theta_2)^3} \end{bmatrix} \\
&= -\begin{bmatrix} -\frac{1}{\theta_2} & -\frac{1}{(\theta_2)^2}E[(X-\theta_1)] \\ -\frac{1}{(\theta_2)^2}E[(X-\theta_1)] & \frac{1}{2}\frac{1}{(\theta_2)^2} - \frac{1}{(\theta_2)^3}E[(X-\theta_1)^2] \end{bmatrix} \\
&= -\begin{bmatrix} -\frac{1}{\theta_2} & 0 \\ 0 & \frac{1}{2}\frac{1}{(\theta_2)^2} - \frac{1}{(\theta_2)^2} \end{bmatrix} \quad \text{(Since } E(X)=\theta_1 \text{ and } E[(X-\theta_1)^2]=\theta_2\text{)} \\
&= -\begin{bmatrix} -\frac{1}{\theta_2} & 0 \\ 0 & -\frac{1}{2}\frac{1}{(\theta_2)^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\theta_2} & 0 \\ 0 & \frac{1}{2}\frac{1}{(\theta_2)^2} \end{bmatrix}
\end{aligned}$$

2 Question 2

Let $X_1, ..., X_n$ be *i.i.d.* with the following PDFs. In each case, find the asymptotic variance of $\sqrt{n}(\hat{\theta}_{MLE} - \theta)$.

(1) $f(x;\theta) = \theta x^{\theta-1}$ for $0 < x < 1$ and zero elsewhere.

(2) $f(x;\theta) = (1/\theta)\exp(-x/\theta)$ for $0 < x$ and zero elsewhere.

2.1 Solution

The density is given by

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1} & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Taking log

$$\log f(x;\theta) = \log \theta x^{\theta-1} = \log\theta + (\theta-1)\log x\,.$$

First and second derivatives are

$$\frac{\partial \log f}{\partial\theta} = \frac{1}{\theta} + \log x, \text{ and } \frac{\partial^2 \log f}{\partial\theta^2} = -\frac{1}{\theta^2}.$$

Then, the Fisher Information is

$$I(\theta) = -E[\frac{\partial^2 \log f}{\partial\theta^2}] = \frac{1}{\theta^2}$$

Therefore, by the theorem

$$\sqrt{n}(\hat{\theta}_{MLE} - \theta) \xrightarrow{d} N(0, \frac{1}{I(\theta)}).$$

Now we have

$$\frac{1}{I(\theta)} = \frac{1}{\frac{1}{\theta^2}} = \theta^2.$$

Thus,

$$\sqrt{n}(\hat{\theta}_{MLE} - \theta) \xrightarrow{d} N(0, \theta^2).$$

(2)
The density is given by

$$f(x;\theta) = \begin{cases} (1/\theta)\exp(-x/\theta) & \text{for } 0 < x \\ 0 & \text{elsewhere} \end{cases}$$

Taking log

$$\log f(x;\theta) = \log[(1/\theta)\exp(-x/\theta)] = -\log\theta - \frac{x}{\theta}$$

First and second derivatives are

$$\frac{\partial \log f}{\partial \theta} = -\frac{1}{\theta} + \frac{x}{\theta^2}.$$

$$\frac{\partial^2 \log f}{\partial \theta^2} = \frac{1}{\theta^2} - \frac{2x}{\theta^3}.$$

Then, the Fisher Information is

$$I(\theta) = -E[\frac{\partial^2 \log f}{\partial \theta^2}] = -E[\frac{1}{\theta^2} - \frac{2X}{\theta^3}] = -\frac{1}{\theta^2} + \frac{2}{\theta^3}E[X]$$

Next,calculating $E[X]$

$$\begin{aligned} E[X] &= \int_0^\infty x(\tfrac{1}{\theta})\exp(-\tfrac{x}{\theta})dx \\ &= \left[-x\exp(-\tfrac{x}{\theta})\right]_0^\infty + \int_0^\infty \exp(-\tfrac{x}{\theta})dx \\ &= [-\theta\exp(-\tfrac{x}{\theta})]_0^\infty \\ &= \theta. \end{aligned}$$

Therefore,

$$I(\theta) = -\frac{1}{\theta^2} + \frac{2}{\theta^3}E[X] = -\frac{1}{\theta^2} + \frac{2}{\theta^3}\theta = \frac{1}{\theta^2}$$

Therefore, by the theorem

$$\sqrt{n}(\hat{\theta}_{MLE} - \theta) \xrightarrow{d} N(0, \frac{1}{I(\theta)}).$$

Now we have

$$\frac{1}{I(\theta)} = \frac{1}{\frac{1}{\theta^2}} = \theta^2.$$

Thus,

$$\sqrt{n}(\hat{\theta}_{MLE} - \theta) \xrightarrow{d} N(0, \theta^2).$$

2.2 Theorem

(Hogg, Mckean, Craig P.325)

Assume $X_1, X_2, \cdots, X_n$ are iid with pdf $f(x;\theta_0)$ for $\theta \in \Omega$ such that the regularity conditions are satisfied. Suppose further that fisher information satisfies $0 < I(\theta_0) < \infty$. Then any consistent sequence of solutions of the mle equations satisfies

$$\sqrt{n}(\hat{\theta}_{MLE} - \theta) \xrightarrow{d} N(0, \frac{1}{I(\theta)}).$$

3 Question 3

Let $X_1, ..., X_{25}$ **be i.i.d.** $N(\mu, 1)$. **We wish to test**

$$\mathbf{H}_0\text{: } \boldsymbol{\mu} = \boldsymbol{\mu}_0$$

$$\mathbf{H}_1\text{: } \boldsymbol{\mu} = \boldsymbol{\mu}_1$$

for some $\mu_1 > \mu_0$. **Derive the best test, i.e., the best critical region, at the 0.05 level.**

3.1 Solution

Following Neyman-Pearson Theorem,

$$C \equiv \left\{ x : \frac{L(\mu_1; x)}{L(\mu_0; x)} \geq k \right\}$$

where k is chosen in such a way that $P[X \in C; H_0] = 0.05$.

$$L(\mu; x) = \prod_{i=1}^{25} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{(x_i - \mu)^2}{2} \right\} = \left(\frac{1}{\sqrt{2\pi}} \right)^{25} \exp\left\{ -\frac{1}{2} \sum_{i=1}^{25} (x_i - \mu)^2 \right\}$$

so

$$\begin{aligned} \frac{L(\mu_1; x)}{L(\mu_0; x)} &= \exp\left\{ -\frac{1}{2} \sum_{i=1}^{25} \left[(x_i - \mu_1)^2 - (x_i - \mu_0)^2 \right] \right\} \\ &= \exp\left\{ -\frac{1}{2} \sum_{i=1}^{25} \left[\mu_1^2 + \mu_0^2 - 2(\mu_1 - \mu_0) x_i \right] \right\} \\ &= \exp\left\{ (\mu_1 - \mu_0) \sum_{i=1}^{25} x_i - \frac{25}{2} (\mu_1^2 + \mu_0^2) \right\} \geq k \end{aligned}$$

Since this ratio is monotonically increasing in $\sum_{i=1}^{25} x_i$,

$\frac{L(\mu_1; x)}{L(\mu_0; x)} \geq k$ iff $\sum_{i=1}^{25} x_i \geq c$

where c is chosen such that $P\left[\sum_{i=1}^{25} x_i \geq c \ \ ; H_0 \right] = 0.05$.

Note that under H_0, $\frac{\overline{X} - \mu_0}{1/\sqrt{25}} \sim N(0, 1)$ where $\overline{X} = \frac{1}{25} \sum_{i=1}^{25} x_i$.

$\sum_{i=1}^{n} x_i \geq c \Longleftrightarrow \overline{X} \geq \frac{c}{25} \Longleftrightarrow \frac{\overline{X} - \mu_0}{1/\sqrt{25}} \geq \frac{\frac{c}{25} - \mu_0}{1/\sqrt{25}} = 1.645$

$\therefore c = 25\mu_0 + 5 * 1.645$

Therefore,

the best critical region is $C = \left\{ x : \ \sum_{i=1}^{25} x_i \geq 25\mu_0 + 5 \times 1.645 \right\}$ ■

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5 Question 4

Let $X_1, ..., X_n$ be i.i.d. random variables such that $X_i \tilde{} N(\mu, 1)$. We would like to test $H_0 : \mu = 0$ against $H_1 : \mu > 0$. A friend of yours suggested a testing strategy where the null hypothesis is rejected if and only if

$$S \equiv \frac{1}{n}\sum_{i=1}^{n} 1(X_i \geq 0) - \frac{1}{2} \geq \frac{1.96}{\sqrt{n}} \times \frac{1}{2}$$

Here, $1(\cdot)$ is an indicator function such that

$$\mathbf{1}(\mathbf{X}_i \geq \mathbf{0}) \equiv \begin{cases} 1, & \text{if } X_i \geq 0 \\ 0, & \text{otherswise} \end{cases}$$

(1) What is the exact probability of Type I error of this test when n=4?
(2) What is the limit of the probability of Type I erro as n$\rightarrow \infty$?

5.1 Solution

(1) What is the exact probability of Type I error of this test when n=4?

Pr [Type I error] = Pr [reject H_0 | H_0] = Pr $[S \geq \frac{1.96}{\sqrt{n}} \times \frac{1}{2} \mid H_0]$
when $n = 4$,

$S \equiv \frac{1}{4}\sum_{i=1}^{n} 1(X_i \geq 0) - \frac{1}{2} \geq \frac{1.96}{\sqrt{4}} \times \frac{1}{2}$
$\Longleftrightarrow \sum_{i=1}^{n} 1(X_i \geq 0) \geq 3.96$
which implies $X_i \geq 0, \ i = 1, 2, 3, 4$.
So the probability of the event "$X_i \geq 0, \ i = 1, 2, 3, 4$,iid " is
$\Pr(X_1 \geq 0|H_0) \times \Pr(X_1 \geq 0|H_0) \times \Pr(X_1 \geq 0|H_0) \times \Pr(X_1 \geq 0|H_0) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

(2) What is the limit of the probability of Type I erro as n$\rightarrow \infty$?

Pr [Type I error] = Pr [reject H_0 | H_0] = Pr $[S \geq \frac{1.96}{\sqrt{n}} \times \frac{1}{2} \mid H_0]$
=Pr $[\frac{1}{n}\sum_{i=1}^{n} 1(X_i \geq 0) - \frac{1}{2} \geq \frac{1.96}{\sqrt{n}} \times \frac{1}{2} \mid H_0]$

$$=\Pr\left[\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{1(X_i \geq 0) - \frac{1}{2}\right\}}{\frac{1}{2}} \geq 1.96 \mid H_0\right]$$

Let $Z_i = 1(X_i \geq 0) \equiv \begin{cases} 1, & \text{if } X_i \geq 0 \text{ with prob. } 0.5 \\ 0, & \text{otherswise with prob. } 0.5 \end{cases}$, that is i.i.d. bernoulli random variable.
$E[Z_i] = \frac{1}{2} < \infty, \quad Var(Z_i) = \frac{1}{4} < \infty$.

By Central Limit Theorem,

$$\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{1(X_i \geq 0) - \frac{1}{2}\right\}}{\frac{1}{2}} = \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(Z_i - \frac{1}{2}\right)}{\frac{1}{2}} \xrightarrow{d} N(0, 1).$$

Accordingly, as $n \rightarrow \infty$,

$$\Pr\left[\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{1(X_i\geq 0)-\frac{1}{2}\right\}}{\frac{1}{2}} \geq 1.96 \mid H_0\right] \to 0.025 \ .$$

cf. Actually, the standard test for this problem is

$$\frac{\overline{X}-0}{1/\sqrt{n}} \geq 1.96.$$

where $\frac{\overline{X}-0}{1/\sqrt{n}} \sim N(0,1)$.

From the above, we can conclude that the a friend's test statistic is asymptotically the same as the standard one. ■