Q-Theory in Continuous Time and No Uncertainty

$$\Pi = \underbrace{\int_t^\infty}_{[sic]} e^{-rt}\Big[Z_t \underbrace{\pi(\overbrace{K_t}^{\text{aggregate } K_t \text{ for whole industry}})}_{\text{productivity of 1 unit of } k_t} k_t - I_t - a\frac{I_t^2}{k_t}\Big] dt$$

$$\text{s.t.} \quad \dot{k}_t = I_t \qquad \delta = 0$$

Current value hamiltonian

$$H_c(k_t, I_t) = Z_t \pi(K_t) k_t - I_t\left(1 + a\frac{I_t}{k_t}\right) + \underbrace{q_t}_{\text{multiplier on constraint}} I_t$$

$$H = \max \int_0^\infty e^{-rt} H_{c_t}\, dt \rightarrow \mu_t = e^{-rt} q_t$$

FOCs:

$$(I_t): \quad \frac{\partial H}{\partial I_t} = 0$$

$$(\mu_t): \quad \frac{\partial H}{\partial \mu} = \dot{k}_t$$

$$(k_t): \quad \frac{\partial H}{\partial k_t} = -\mu$$

$$\Rightarrow (I_t): \quad 1 + \frac{2aI_t}{k_t} = q_t$$

$$(\mu_t): \quad I_t = \dot{k}_t$$

$$(k_t): \quad e^{-rt}\left[Z_t \pi(K_t) + a\left(\frac{I_t}{k_t}\right)^2\right] = -\overbrace{\left[e^{-rt}\dot{q}_t - r q_t e^{-rt}\right]}^{\dot{\mu}}$$

$$\Rightarrow Z_t \pi(K_t) + a\left(\frac{I_t}{k_t}\right)^2 = r q_t - \dot{q}_t$$

<u>TVC:</u> $\lim_{t\to\infty} e^{-rt} q_t k_t = 0$

Solving for the marginal q_t:

- $e^{-rt}\left[Z_t \pi(K_t) + a\left(\frac{I_t}{k_t}\right)^2\right] = e^{-rt}\left[r q_t - \dot{q}_t\right]$
- integrate over t and T:
- $\int_t^T e^{-r\tau}\left[Z_\tau \pi(K_\tau) + a\left(I_\tau/k_\tau\right)^2\right]d\tau = \int_t^T \overbrace{e^{-r\tau}(r q_\tau - \dot{q}_\tau)}^{-\dot{\mu}} d\tau$

$$= -\left(e^{-rt} q_t + B\right)_t^T$$

$$\Rightarrow -e^{-rT} q_T + e^{-rt} q_t = \int_t^T (\cdot)\, d\tau$$

$$e^{-rt} q_t = \int_t^T (\cdot)\, d\tau + \underbrace{e^{-rT} q_T}_{\to 0 \text{ as } T\to\infty}$$

$$\Rightarrow 1 + 2a\frac{I_t}{K_t} = q_t$$

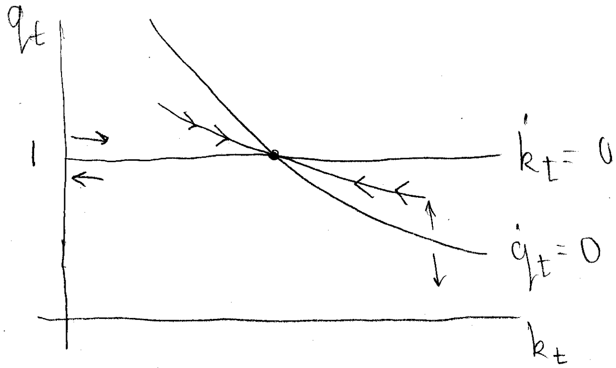
$$\underbrace{q_t}_{\substack{\text{shadow value} \\ \text{of one unit} \\ \text{of } K_t.}} = \int_t^\infty e^{-r(\tau-t)}\left[Z_t \pi(K_\tau) + a\left(\frac{I_\tau}{K_\tau}\right)^2\right]d\tau$$

System of differential equations

- $1+2a\frac{I_t}{k_t} = q_t \Rightarrow 1+2a\frac{\dot{k}_t}{k_t} = q_t$ (1)

- $Z_t\Pi(K_t) + a\left(\frac{I_t}{k_t}\right)^2 - rq_t = \dot{q}_t$ (2)

(1) $\Rightarrow \frac{\dot{k}_t}{k_t} = \frac{q_t-1}{2a}$; $\dot{k}_t = 0 \Rightarrow q_t = 1$



(2) $\Rightarrow$ when $\dot{q}_t = 0$,

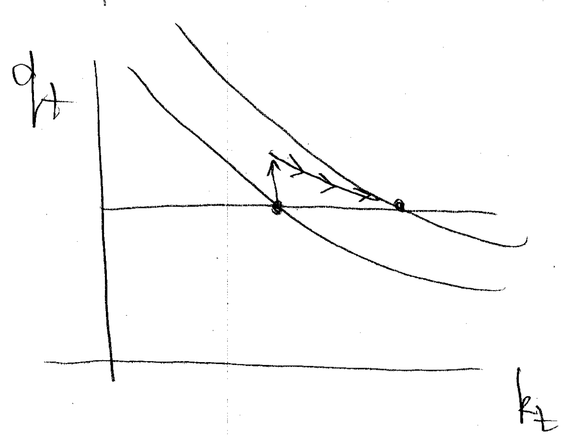
$$Z_t\Pi(K_t) + a\left(\frac{q_t-1}{2a}\right)^2 = rq_t$$

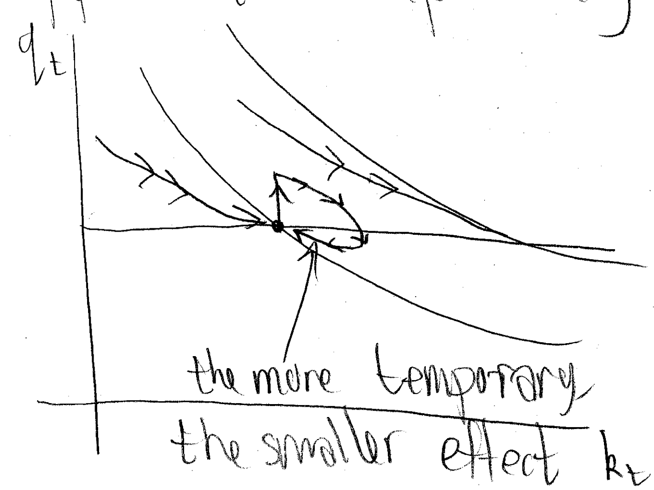
$$\Rightarrow \frac{dq_t}{dK_t} = \frac{Z_t\Pi'(K_t)}{\underbrace{r - \frac{q_t-1}{2a}}_{\text{assume } >0}} < 0$$

- since in equilibrium, $k_t = K_t$

Comparative Dynamics

Suppose $Z_t \uparrow$ permantly; Suppose $Z_t \uparrow$ temporarily





Investment Tax credit

- Fraction θ of the price of investment
- Rebate applies to investment only. Not to revenue

$$\hat{H} = Z_t \Pi(K_t) k_t - I_t(1-\theta) - a\frac{I_t^2}{k_t} + q_t I_t$$

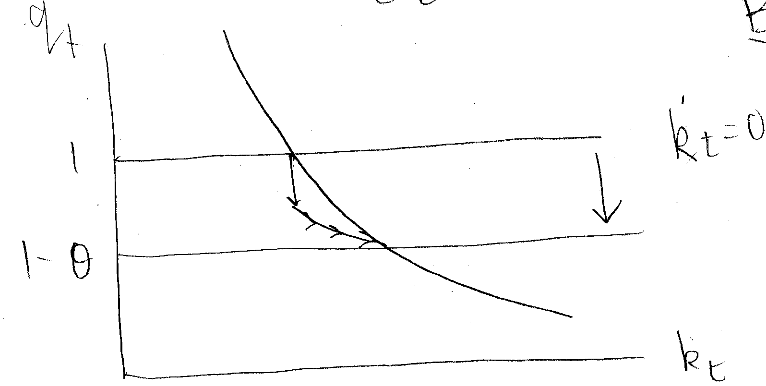
FOCs:

$$(I_t): -(1-\theta) - 2a\frac{I_t}{k_t} + q_t = 0 \qquad (1)$$

$$\Rightarrow q_t + \theta = 1 + 2a\frac{I_t}{k_t}$$

$$\dot{q}_t = r q_t - \left[Z_t \Pi(K_t) + a\left(\frac{q_t+\theta}{2a} - 1\right)^2\right]$$

$$(1) \Rightarrow \frac{\dot{k}_t}{k_t} = \frac{q_t+\theta-1}{2a} \qquad \dot{k}_t = 0 \Rightarrow q_t + \theta = 1$$

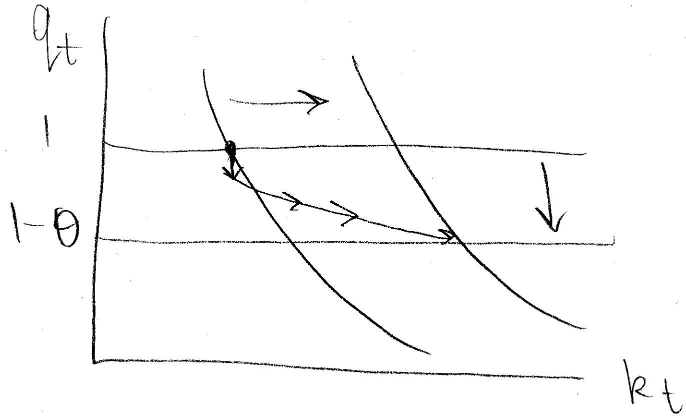


<u>Both</u> $\dot{q}_t = 0$ and $\dot{k}_t = 0$ equations change

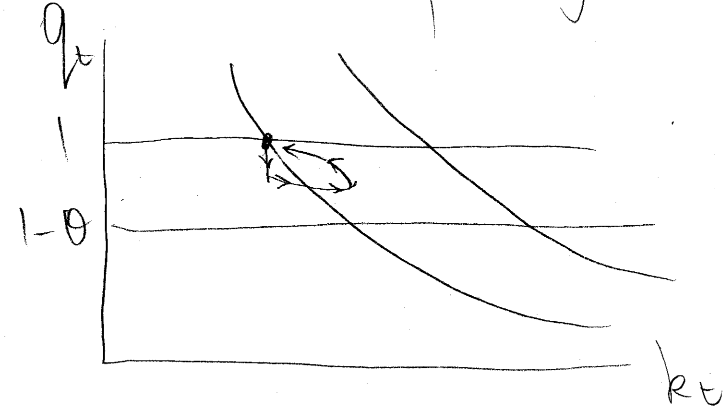
$\dot{q}_t = 0 \Rightarrow \quad r q_t - \frac{a}{4a^2}(q_t - 1 + \theta)^2 = z_t \pi(k_t)$

$\Rightarrow \quad r q_t - \frac{1}{4a}(q_t - 1)^2 = z_t \pi(k_t) + \frac{\theta^2}{4a} + \frac{1}{2a}\theta(q_t - 1)$

$\Rightarrow$ this line shifts to the right



What if temporary?



More temporary $\Rightarrow$ greater boom.