Structural VARs

W_0 matrix

$$\underline{Y}_t = \begin{bmatrix} Y_{1t} \\ \vdots \\ Y_{kt} \end{bmatrix} \Big\} \text{ endogenous variables}$$

- $W_0 \underline{Y}_t = W_1 \underline{Y}_{t-1} + \ldots + W_p \underline{Y}_{t-p} + u_t$
- $E[u_t] = 0$
- $E[u_t u_s'] = 0 \quad s \neq t$
- $E[u_t u_t'] = \Omega$, $E[u_{it} u_{jt}] = 0$

Cannot use OLS, because we have endogeneity

$$\underbrace{\underline{Y}_t}_{\text{reduced form VAR}} = W_0^{-1} W_1 \underline{Y}_{t-1} + W_0^{-1} W_2 \underline{Y}_{t-2} + \ldots + W_0^{-1} W_p \underline{Y}_{t-p} + W_0^{-1} u_t$$

$$\equiv A_1 \underline{Y}_{t-1} + \ldots + A_p \underline{Y}_{t-p} + e_t$$

- $E[e_t] = E[W_0^{-1} u_t] = W_0^{-1} E[u_t] = 0$
- $E[e_t e_t'] = E[W_0^{-1} u_t u_t' (W_0^{-1})']$
  $= W_0^{-1} \Omega (W_0^{-1})'$
- $E[e_t e_s'] = W_0^{-1} \underbrace{E[u_t u_s']}_{=0} (W_0^{-1})' = 0$

but $E[e_{it} e_{jt}] \neq 0$

$$Y_t = A_1 Y_{t-1} + \dots + A_p Y_{t-p} + \dots + e_t$$
$$= A_1 L Y_t + \dots + A_p L^p Y_{t-p} + \dots + e_t$$

$$\Rightarrow (I - A_1 L - \dots - A_p L^p - \dots) Y_t = e_t$$

$\Rightarrow Y_t = A(L) e_t$ MA(∞) process

also, $Y_t = B(L) Y_{t-1} + e_t$ VAR(1) process

since $Y_t = A_1 Y_{t-1} + A_2 L Y_{t-1} + \dots + A_p L^{p-1} Y_{t-p} + \dots + e_t$
$$= (A_1 + A_2 L + \dots + A_p L^{p-1} + \dots) Y_{t-1} + e_t$$

Covariance stationarity

$E[Y_t]$ independent of t
$Var(Y_t)$ constant
$E[Y_t, Y_s]$ is a function of $(t-s)$.

If $Y_t = a Y_{t-1} + \beta Y_{t-2} + \varphi_1 e_t + \varphi_2 e_{t-1}$ ARMA(2,1)

(*) Read first chapter of Hamilton

$$\left.\begin{aligned} Y_t &= C(L)\eta_t \\ \eta_t &= Q e_t \end{aligned}\right\} \text{ want this}$$

$A(L)^{-1} Y_t = e_t$

$\Rightarrow \underbrace{Q\, A(L)^{-1}}_{\equiv C(L)^{-1}} Y_t = Q e_t = \eta_t$

$\Rightarrow \quad Y_t = C(L)\, \eta_t$

want orthogonal errors: $E[\eta_{it}\eta_{jt}] = 0 \quad \forall i \neq j$

- $E[\eta_t \eta_t'] = E[Q e_t e_t' Q'] = Q \underbrace{E[e_t e_t']}_{\equiv \Sigma = W_0^{-1}\Omega(W_0^{-1})'} Q'$

$= Q \Sigma Q' = I$

Want $\Sigma = Q^{-1}(Q^{-1})'$

- can use the Choleski decomposition:
  $A = LL'$, L lower triangular
- This is not unique, because any R s.t.
  $RR' = I$ gives us: $RQ\Sigma Q'R'$
  $= RIR' = RR' = I$

$C(L)$ restrictions to pick "right" Q matrix

- $C(L) = A(L)\, Q^{-1}$
- short-run restrictions: $C(0)$: $Y_t = C_0\eta_t + C_1\eta_{t-1} +$

$$C_0 = \begin{bmatrix} C_{11}^0 & C_{12}^0 \\ C_{21}^0 & C_{22}^0 \end{bmatrix}$$

want to restrict some of these elements to be zero (ie $C_{12}^0 = 0$)

- want this lower triangular

$\Rightarrow C(0) = A(0)Q^{-1}$

$\Rightarrow A(0)^{-1}C(0) = Q^{-1}$

also, $C(0)C(0)' = A(0)Q^{-1}(Q^{-1})'A(0)'$

Long-run restrictions: Blanchard, Quah (1989)

$$\begin{bmatrix} \Delta Y_t \\ \Delta M_t \end{bmatrix} = C(L)\eta_t$$

- impose restrictions on $C(1)$
  - call these restrictions (LR)
  - $C(L) = \sum_{j=0}^{\infty} c_j L^j$
  - $C_0 + C_1 L\eta_t + C_2 L^2 + \cdots$

Read Blanchard and Quah.