Chapter 4: (next time: endogenous growth)

Given $\{R_t\}_{t=0}^{\infty}$, define for $q_0 > 0$,

$$q_t = \frac{q_0}{(1+R_1)\cdots(1+R_t)}$$ price of pd t consumption at pd. 0.

BC: $c_t + a_{t+1} \leq (1+R_t)a_t + y_t \quad \forall t$

$$\Rightarrow \sum_{t=0}^{T} q_t c_t + q_T \cdot a_{T+1} \leq q_0(1+R_0)a_0 + \sum_{t=0}^{T} q_t y_t$$

No ponzi: $\lim_{T\to\infty} q_T a_{T+1} = 0$

$$\Rightarrow \sum_{t=0}^{\infty} q_t c_t \leq q_0(1+R_0)a_0 + \sum_{t=0}^{\infty} q_t y_t$$

(cf static problem with consumption in different pds as different goods)

Can also recover per-period BC by solving recursively for a_{t+1}.

If have borrowing constraints, cannot necessarily go from intertemporal bc to per-period bc. (don't have complete markets.)

With this equivalence between per period bcs and intertemporal bc, can think of 2 mkt structures.

- Trade everything at the beginning of time
- Spot mkts in c_t, a_{t+1}.

Dynamic vs. static mkt structure.

$$BC \Leftrightarrow \sum_{t=0}^{T} q_t c_t \leq q_0 x_0$$

- $x_0 = (1+R_0)a_0 + h_0,\quad \overbrace{h_0 \equiv \sum_{t=0}^{T} \frac{q_t}{q_0} y_t}^{\text{human wealth}}$

effective wealth of the household

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t} \quad \sum_{t=0}^{\infty} q_t c_t \leq q_0 x_0$$

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda\left[q_0 x_0 - \sum_{t=0} q_t c_t\right]$$

- the fact that we now only need one Lagrange multipliers arises from the fact that w/ complete mkts can equate shadow prices across prices.

- When interest rate equals discount rate, you equalize MU across periods. Usually, this implies equalizing consumption levels over time.

Arrow-Debreu markets:

- at each t, draw s_t w/ finite support
  Let $s^t = \{s_0, s_1, \ldots, s_t\}$
- at $t=0$, open complete mkts. If T pds, there are $1+S+S^2+\ldots+S^T$ mkts, where S is the number of elements in support of s_t. (assumed constants)

- $q(s^t)$ be pd 0 price of unit of consumption in pd t and event s^t
  - $w(s^t)$ - corresponding wage.

$$\max \sum_t \sum_{s^t} \beta^t \pi(s^t) U(c^j(s^t), z^j(s^t))$$

$$\sum_t \sum_{s^t} [q(s^t) \cdot c^j(s^t) + q(s^t) w(s^t) \cdot z^j(s^t)] \leq q_0 \cdot \bar{x}_0^j$$

$$\bar{x}_0^j = (1+R_0) a_0 + \bar{h}_0^j$$

$$\bar{h}_0^j = \sum_{t=0}^{\infty} \frac{q(s^t)}{q_0} [w(s^t) \cdot \bar{z} - T^j(s^t)]$$

- The ability to freely transfer consumption across periods and states (complete mkts) gives us the same equivalence as before.

- $j \in \{1, \dots, J\}$
- $e_t^j(s^t)$ - endowment for j at t in state s^t.

$\Rightarrow \quad \sum_j e_t^j(s^t) = \bar{e}_t(s^t)$ (allow for agg. risk)

- Facing A-D (arrow Debreu) prices $q_t(s^t)$
  - as of pd 0, $q_t(s^t)$ is price of consumption at t in state s^t.

Consumer wants to, st. $q_t(s^t)$,

$$\max \sum_t \sum_{s^t} \beta^t \pi(s^t)\, u(c_t^j(s^t))$$

$$\text{s.t. } \sum_t \sum_{s^t} q_t(s^t)\, c_t^j(s^t) \leq \sum_t \sum_{s^t} q_t(s^t)\, e_t^j(s^t)$$

FOCs:

$$\beta^t \pi(s^t)\, u'(c_t^j(s^t)) = \lambda q_t(s^t) \qquad \forall t, \forall s^t, \forall j$$

Normalize $q_0 = 1$

$$\Rightarrow \quad u'(c_0^j) = \lambda \qquad \forall j$$

$$\Rightarrow \frac{q_t(s^t)}{q_0} = \frac{\beta^t \pi(s^t) u'(c_t^j(s^t))}{u'(c_0^j)} = MRS_{c_t^j, c_0^j} \quad (1)$$

assume $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$

$\Rightarrow$ (1) becomes

$$(2) \quad q_t(s^t) = \beta^t \pi(s^t) \left(\frac{c_t^j(s^t)}{c_0^j} \right)^{-\gamma} \quad \forall j$$

- consumption "growth" constant across households.

Resource constraint:

$$\sum_j c_t^j(s^t) \overset{\text{by monotonicity of b.c.}}{=} \sum_j e_t^j(s^t) = \bar{e}_t(s^t) \; \forall t, \forall s^t.$$

In equilibrium, $\frac{c_t^j(s^t)}{c_0^j} = \frac{\bar{e}_t(s^t)}{\bar{e}_0}$ by feasibility

Then (2) becomes:

$$q_t(s^t) = \beta^t \pi(s^t) \left(\frac{\bar{e}_t(s^t)}{\bar{e}_0} \right)^{-\gamma}$$

$$= \beta^t \pi(s^t) \left[\frac{u'(\bar{e}_t(s^t))}{u'(\bar{e}_0)} \right]$$

Thus, prices are pinned down by aggregate endowment.

Suppose $\{d_t(s^t)\}$ pays at $t=3 \begin{cases} 5 \text{ if rains} \\ 3 \text{ if not rains} \end{cases}$

$f_3(\text{rain})$ = pays at $t=3$, 1 if rains

$f_3(\text{not rain})$ = pays at $t=3$, 1 if not rains

Then $5 f_t(s_1^t) + 3 f_t(s_2^t) \Leftrightarrow \{d_t(s^t)\}$

Thus, $\text{Price}(\underbrace{\{d_t(s^t)\}}_{\text{asset}}) = \sum_t \sum_{s^t} q_t(s^t) d_t(s^t)$