Theorem: Let F be concave, differentiable, $X \subseteq \mathbb{R}^n_+$, $F_x \geq 0$. Then, if EE and TC hold, then x^* (the sequence) is optimal.

Pf: Let $D = \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \left[F(x_t^*, x_{t+1}^*) - F(x_t, x_{t+1}) \right]$

$$(1) \quad \geq \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \underbrace{\left[\overbrace{F_x(x_t^*, x_{t+1}^*)}^{\equiv F_{x,t}^*}(x_t^* - x_t) + \overbrace{F_y(x_t^*, x_{t+1}^*)}^{F_{y,t}^*}(x_{t+1}^* - x_t) \right]}$$

since a concave fcn lies below its tangent.

since $x_0^* - x_0 = 0$ and

Euler equation: $F_{y,t}^* + \beta F_{x,t+1}^* = 0$, we have that

$$(1) = \lim_{T\to\infty} \beta^T \cdot F_{y,t}^* (x_{T+1}^* - x_{T+1})$$

$$= -\lim_{T\to\infty} \beta^{T+1} F_{x,T+1}^* (x_{T+1}^* - x_{T+1}) \quad \text{by Euler equation.}$$

$$= -\lim_{T\to\infty} \beta^{T+1} F_{x,T+1}^* x_{T+1}^* + \underbrace{\lim_{T\to\infty} \beta^{T+1} \underbrace{F_{x,T+1}^*}_{\geq 0} \underbrace{x_{T+1}}_{\geq 0}}_{\geq 0}$$

$$\geq -\lim_{T\to\infty} \beta^{T+1} F_{x,T+1}^* x_{T+1}^* = 0$$

$$\Rightarrow \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t F(x_t^*, x_{t+1}^*) \geq \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1})$$

$\Rightarrow (x_t^*)_{t=0}^{\infty}$ is optimal. $\square$

Inheritability

1] $F(\cdot, y)$ increasing and $\Gamma(x) \subseteq \Gamma(x')$ if $x' \geq x$

$\Rightarrow V^*(x)$ is increasing in x.

Define $A(\Gamma) = \{(x,y) : y \in \Gamma(x)\}$ to be the graph of Γ.

2] If F is concave and $A(\Gamma)$ is a convex set

$(x, x', y \in \Gamma(x), y' \in \Gamma(x') \Rightarrow \forall \alpha \in [0,1]$,

$y'' = \alpha y + (1-\alpha) y' \in \Gamma(x'') = \Gamma(\alpha x + (1-\alpha) x'))$, then

$V^*(x)$ is concave in x

3] If F is differentiable, can we prove that V is differentiable? Need more conditions for this to be true.

- e.g. F concave and A is convex, Γ continuous

Proof: $V^*(x) = F(x, g(x)) + \beta V^*(g(x)) \geq W(x) = F(x, g(x_0)) + \beta V^*(g(x_0))$;

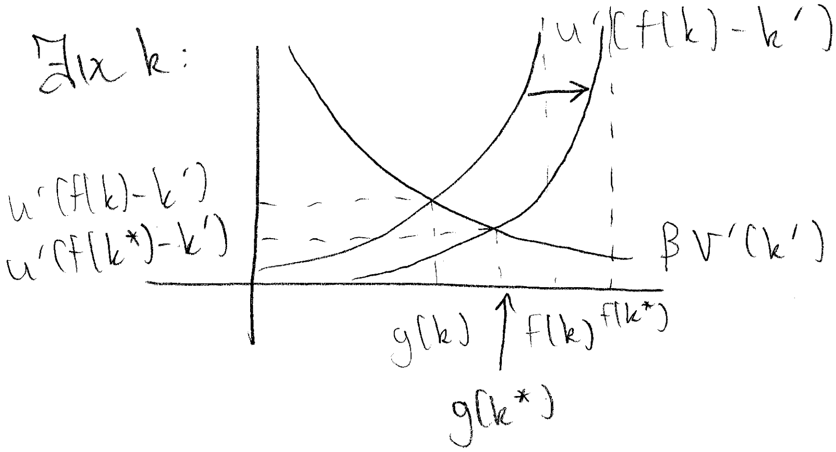
- $g(x_0) \in \Gamma(x)$, $x \in N_{x_0}$

$\Rightarrow W'(x_0) = F_x(x_0, g(x_0))$

$\Rightarrow V^{*\prime}(x_0) = W'(x_0) = F_x(x_0, g(x_0))$.

$u'(f(k)-k') = \beta v^{*\prime}(k')$

Fix k:



- Increase k to k*

- Since $u'(f(k)-k') \geq u'(f(k^*)-k')$, by concavity, $c \geq c^*$

- Further $g(k^*) \geq g(k)$

$\Rightarrow$ consume more and save more.