14.451: Macroeconomic Theory I

Overview of Key Results: Chapter 2, Acemoglu

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1 Propositions, Definitions, etc.

Assumption 1. (Continuity, Differentiability, Positive and Diminishing Marginal Products, and Constant Returns to Scale) The production function $F : \mathbb{R}_+^3 \to \mathbb{R}_+$ is twice continuously differentiable in K and L, and satisfies

$$F_K(K,L,A) \equiv \frac{\partial F(K,L,A)}{\partial K} > 0, \qquad F_L(K,L,A) \equiv \frac{\partial F(K,L,A)}{\partial L} > 0,$$
$$F_{KK}(K,L,A) \equiv \frac{\partial^2 F(K,L,A)}{\partial K^2} < 0, \quad F_{LL}(K,L,A) \equiv \frac{\partial^2 F(K,L,A)}{\partial L^2} < 0.$$

Morever, F exhibits constant returns to scale in K and L.

Definition 2.1. Let $z \in \mathbb{R}^K$ for some $K \geq 1$. The function $g(x,y,z)$ is **homogeneous of degree** m in $x \in \mathbb{R}$ and $y \in \mathbb{R}$ if and only if

$$g(\lambda x, \lambda y, z) = \lambda^m g(x,y,z) \text{ for all } \lambda \in \mathbb{R}_+ \text{ and } z \in \mathbb{R}^K.$$

Theorem 2.1 (Euler's Theorem) Suppose that $g : \mathbb{R}^{K+2} \to \mathbb{R}$ is continuously differentiable in $x \in \mathbb{R}$ and $y \in \mathbb{R}$, with partial derivatives denoted by g_x and g_y and is homogeneous of degree m in x and y. Then

$$mg(x,y,z) = g_x(x,y,z)\,x + g_y(x,y,z)\,y \text{ for all } x \in \mathbb{R},\ y \in \mathbb{R} \text{ and } z \in \mathbb{R}^K.$$

Morever, $g_x(x,y,z)$ and $g_y(x,y,z)$ are themselves homogeneous of degree $m-1$ in x and y.

Proposition 2.1. Suppose Assumption 1 holds. Then in the equilibrium of the Solow growth model, firms make no profits, and in particular,

$$Y(t) = w(t)\,L(t) + R(t)\,K(t).$$

Assumption 2. (Inada Conditions) F satisfies the Inada conditions

$$\lim_{K\to 0} F_K(K,L,A) = \infty \text{ and } \lim_{K\to\infty} F_K(K,L,A) = 0 \text{ for all } L > 0 \text{ and all } A$$
$$\lim_{L\to 0} F_L(K,L,A) = \infty \text{ and } \lim_{L\to\infty} F_L(K,L,A) = 0 \text{ for all } K > 0 \text{ and all } A.$$

If a production function F satisfies assumption 1 and assumption 2, we say that it is a **Neoclassical production function.**

Definition 2.2 In the basic Solow model for a given sequence of $\{L(t), A(t)\}_{t=0}^{\infty}$ and an initial capital stock $K(0)$, an **equilibrium path** is a sequence of capital stocks, output levels, consumption levels, wages, and rental rates $\{K(t), Y(t), C(t), w(t), R(t)\}_{t=0}^{\infty}$ such that $K(t)$ satisfies (2.11), $Y(t)$ is given by (2.1), $C(t)$ is given by (2.10), and $w(t)$ and $R(t)$ are given by (2.5) and (2.6).

Definition 2.3. A **steady-state equilibrium without technological progress and population growth** is an equilibrium path in which $k(t) = k^*$ for all t.

Proposition 2.2. Consider the basic Solow growth model and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where the capital-labor ration $k^* \in (0, \infty)$ is given by (2.17), per capita output is given by

$$y^* = f(k^*)$$

and per capita consumption is given by

$$c^* = (1 - s) f(k^*).$$

Proposition 2.3. Suppose Assumptions 1 and 2 hold and $f(k) = a\tilde{f}(k)$. Denote the steady-state level of the capital-labor ratio by $k^*(a, s, \delta)$ and the steady-state level of output by $y^*(a, s, \delta)$ when the underlying parameters are a, s and δ. Then we have

$$\begin{aligned} \frac{\partial k^*(a,s,\delta)}{\partial a} &> 0, \ \frac{\partial k^*(a,s,\delta)}{\partial s} > 0 \text{ and } \frac{\partial k^*(a,s,\delta)}{\partial \delta} < 0 \\ \frac{\partial y^*(a,s,\delta)}{\partial a} &> 0, \ \frac{\partial y^*(a,s,\delta)}{\partial s} > 0 \text{ and } \frac{\partial y^*(a,s,\delta)}{\partial \delta} < 0. \end{aligned}$$

Proposition 2.4. In the basic Solow growth model, the highest level of consumption is reached for s_{gold}, with the corresponding steady state capital level k^*_{gold} such that

$$f'\left(k^*_{gold}\right) = \delta.$$

Definition 2.4. A steady state is (locally) **asymptotically stable** if there exists an open set $B(\mathbf{x}^*) \ni \mathbf{x}^*$ such that for any solution $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ to (2.23) with $\mathbf{x}(0) \in B(\mathbf{x}^*)$, we have that $\mathbf{x}(t) \to \mathbf{x}^*$. Moreover, $\mathbf{x}^*$ is globally asymptotically stable if for all $\mathbf{x}(0) \in \mathbb{R}^n$, for any solution $\{\mathbf{x}(t)\}_{t=0}^{\infty}$, we have that $\mathbf{x}(t) \to \mathbf{x}^*$.

Theorem 2.2. Consider the following linear difference equation system

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}$$

with initial value $\mathbf{x}(0)$, where $\mathbf{x}(t) \in \mathbb{R}^n$ for all t, $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{b}$ is an $n \times 1$ column vector. Let $\mathbf{x}^*$ be the steady state of the difference equation given by $\mathbf{A}\mathbf{x}^* + \mathbf{b} = \mathbf{x}^*$. Suppose that all of the eigenvalues of $\mathbf{A}$ are strictly inside the unit circle in the complex plane. Then the steady state of the difference equation (2.24), $\mathbf{x}^*$, is globally asymptotically stable, in the sense that starting from any $\mathbf{x}(0) \in \mathbb{R}^n$, the unique solution $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ satisfies $\mathbf{x}(t) \to \mathbf{x}^*$.

Theorem 2.3. Consider the following nonlinear autonomous system

$$\mathbf{x}(t+1) = \mathbf{G}[\mathbf{x}(t)]$$

with initial value $\mathbf{x}(0)$, where $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$. Let $\mathbf{x}^*$ be a steady state of this system, i.e., $\mathbf{G}(\mathbf{x}^*) = \mathbf{x}^*$, and suppose that $\mathbf{G}$ is continuously differentiable at $\mathbf{x}^*$. Define

$$\mathbf{A} \equiv \nabla\mathbf{G}(\mathbf{x}^*),$$

and suppose that all of the eigenvalues of $\mathbf{A}$ are strictly inside the unit circle. Then the steady state of the difference equation (2.25) $\mathbf{x}^*$ is locally asymptotically stable, in the sense that there exists an open neighborhood of $\mathbf{x}^*$, $\mathbf{B}(\mathbf{x}^*) \subset \mathbb{R}^n$ such that starting from any $\mathbf{x}(0) \in \mathbf{B}(\mathbf{x}^*)$, we have $\mathbf{x}(t) \to \mathbf{x}^*$.

Corollary 2.1. Let $x(t), a, b \in \mathbb{R}$. Then the unique steady state of the linear difference equation $x(t+1) = ax(t) + b$ is globally asymptotically stable (in the sense that $x(t) \to x^* = \frac{b}{1-a}$) if $|a| < 1$.

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function, differentiable at the steady state x^*, defined by $g(x^*) = x^*$. Then, the steady state of the nonlinear difference equation $x(t+1) = g(x(t))$, x^*, is locally asymptotically stable if $|g'(x^*)| < 1$. Moreover, if $|g'(x)| < 1$ for all $x \in \mathbb{R}$, then x^* is globally asymptotically stable.

Proposition 2.5. Suppose that Assumptions 1 and 2 hold, then the steady-state equilibrium of the Solow growth model described by the difference equation (2.16) is globally asymptotically stable, and starting from any $k(0) > 0$, $k(t) \to k^*$.

Proposition 2.6. Suppose that Assumptions 1 and 2 hold, and $k(0) < k^*$, then $\{w(t)\}_{t=0}^{\infty}$ is an increasing sequence and $\{R(t)\}_{t=0}^{\infty}$ is a decreasing sequence. If $k(0) > k^*$, the opposite results apply.

Definition 2.5. In the basic Solow model in continuous time with population growth rate n, no technological progress and an initial capital stock $K(0)$, an **equilibrium path** is a sequence of capital stocks, labor, output levels, consumption levels, wages, and rental rates $[K(t), L(t), Y(t), C(t), w(t), R(t)]_{t=0}^{\infty}$ such that $K(t)$ satisfies (2.32), $L(t)$ satisfies (2.31), $Y(t)$ is given by (2.1), $C(t)$ is given by (2.10), and $w(t)$ and $R(t)$ are given by (2.5) and (2.6).

Proposition 2.7. Consider the basic Solow growth model in continuous time and suppose Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where the capital-labor ratio is equal to $k^* \in (0, \infty)$ and is given by (2.33), per capita output is given by

$$y^* = f(k^*)$$

and per capita consumption is given by

$$c^* = (1-s) f(k^*).$$

Proposition 2.8. Suppose Assumptions 1 and 2 hold and $f(k) = a\tilde{f}(k)$. Denote the steady-state equilibrium level of the capital-labor ratio by $k^*(a, s, \delta, n)$ and the steady-state level of output by $y^*(a, s, \delta, n)$ when the underlying parameters are given by a, s and δ. Then we have

$$\begin{aligned}
\frac{\partial k^*(a,s,\delta,n)}{\partial a} &> 0, \quad \frac{\partial k^*(a,s,\delta,n)}{\partial s} > 0, \quad \frac{\partial k^*(a,s,\delta,n)}{\partial \delta} < 0 \text{ and } \frac{\partial k^*(a,s,\delta,n)}{\partial n} < 0 \\
\frac{\partial y^*(a,s,\delta,n)}{\partial a} &> 0, \quad \frac{\partial y^*(a,s,\delta,n)}{\partial s} > 0, \quad \frac{\partial y^*(a,s,\delta,n)}{\partial \delta} < 0 \text{ and } \frac{\partial y^*(a,s,\delta,n)}{\partial n} < 0.
\end{aligned}$$

Theorem 2.4. Consider the following linear differential equation system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) = \mathbf{b}$$

with initial value $\mathbf{x}(0)$, where $\mathbf{x}(t) \in \mathbb{R}^n$ for all t, $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{b}$ is an $n \times 1$ column vector. Let $\mathbf{x}^*$ be the steady state of the system given by $\mathbf{A}\mathbf{x}^* + \mathbf{b} = \mathbf{0}$. Suppose that all of the eigenvalues of $\mathbf{A}$ have negative real parts. Then the steady state of the differential equation (2.34) $\mathbf{x}^*$ is globally asymptotically stable, in the sense that starting from any $\mathbf{x}(0) \in \mathbb{R}^n$, $\mathbf{x}(t) \to \mathbf{x}^*$.

Theorem 2.5. Consider the following nonlinear autonomous differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{G}[\mathbf{x}(t)]$$

with initial value $\mathbf{x}(0)$, where $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$. Let $\mathbf{x}^*$ be a steady state of this system, i.e. $\mathbf{G}(\mathbf{x}^*) = 0$, and suppose that $\mathbf{G}$ is continuously differentiable at $\mathbf{x}^*$. Define

$$\mathbf{A} \equiv \nabla \mathbf{G}(\mathbf{x}^*),$$

and suppose that all of the eigenvalues of $\mathbf{A}$ have negative real parts. Then the steady state of the differential equation (2.35) $\mathbf{x}^*$ is locally asymptotically stable, in the sense that there exists an open neighborhood of $\mathbf{x}^*$, $\mathbf{B}(\mathbf{x}^*) \subset \mathbb{R}^n$ such that starting from any $\mathbf{x}(0) \in \mathbf{B}(\mathbf{x}^*)$, we have $\mathbf{x}(t) \to \mathbf{x}^*$.

Corollary 2.2. Let $x(t) \in \mathbb{R}$, then the steady state of the linear differential equation $\dot{x}(t) = ax(t)$ is globally asymptotically stable (in the sense that $x(t) \to 0$) if $a < 0$.

Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and differentiable at x^* where $g(x^*) = 0$. Then, the steady state of the nonlinear differential equation $\dot{x}(t) = g(x(t))$, x^*, is locally asymptotically stable if $g'(x^*) < 0$.

Theorem 2.6. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function and suppose that there exists a unique x^* such that $g(x^*) = 0$. Morever, suppose $g(x) < 0$ for all $x > x^*$ and $g(x) > 0$ for all $x < x^*$. Then the steady state of the nonlinear differential equation $\dot{x}(t) = g(x(t))$, x^*, is globally asymptotically stable, i.e., starting with any $x(0)$, $x(t) \to x^*$.

Proposition 2.9. Suppose that Assumptions 1 and 2 hold, then the basic Solow growth model in continuous time with constant population growth and no technological change is globally asymptotically stable, and starting from any $k(0) > 0$, $k(t) \to k^*$.

Proposition 2.10. Consider the Solow growth model with the production function (2.38) and suppose that $sA - \delta - n > 0$. Then in equilibrium, there is sustained growth of output per capita at the rate $sA - \delta - n$. In particular, starting with a capital-labor ratio $k(0) > 0$, the economy has

$$k(t) = \exp((sA - \delta - n)t)\, k(0)$$

and

$$y(t) = \exp((sA - \delta - n)t)\, Ak(0).$$

Proposition 2.11. (Uzawa) Consider a growth model with a constant returns to scale aggregate production function

$$Y(t) = F\left[K(t), L(t), \tilde{A}(t)\right],$$

with $\tilde{A}(t)$ representing technology at time t and aggregate resource constraint

$$\dot{K}(t) = Y(t) - C(t) - \delta K(t).$$

Suppose that there is a constant growth rate of population, i.e., $L(t) = \exp(nt)\, L(0)$ and that there exists an asymptotic path where output, capital and consumption grow at constant rates, i.e., $\dot{Y}(t)/Y(t) = g_Y$, $\dot{K}(t)/K(t) = g_K$ and $\dot{C}(t)/C(t) = g_C$. Suppose finally that $g_K + \delta > 0$. Then,

(1) $g_Y = g_K = g_C$; and

(2) asymptotically, the aggregate production function can be represented as:

$$Y(t) = \tilde{F}[K(t), A(t)L(t)],$$

where

$$\frac{\dot{A}(t)}{A(t)} = g = g_Y - n.$$

Corollary 2.3. Under the assumptions of Proposition 2.11, if an economy has an asymptotic path with constant growth of output, capital and consumption, then asymptically technological progress can be represented as Harrod neutral (purely labor augmenting).

Corollary 2.4. Under the conditions of Proposition 2.11, if factor markets are competitive, then asymptotic factor shares are constant, i.e. as $t \to \infty$, $\alpha_L(t) \to \alpha_L^*$ and $\alpha_K(t) \to \alpha_K^*$.

Proposition 2.12. Consider the basic Solow growth model in continuous time, with Harrod-neutral technological progress at the rate g and population growth at the rate n. Suppose that Assumptions 1 and 2 hold, and define the effective capital-labor ratio as in (2.43). Then there exists a unique steady state (balanced growth path) equilibrium where the effective capital-labor ratio is equal to $k^* \in (0, \infty)$ and is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta + g + n}{s}.$$

Per capita output and consumption grow at the rate g.

Proposition 2.13. Suppose Assumptions 1 and 2 hold and let $A(0)$ be the initial level of technology. Denote the balanced growth path level of effective capital-labor ratio by $k^*(A(0), s, \delta, n)$ and the level of output per capita by $y^*(A(0), s, \delta, n, t)$ (the latter is a function of time since it is growing over time). Then we have

$$\begin{aligned}
\frac{\partial k^*(A(0), s, \delta, n)}{\partial A(0)} &= 0, \ \frac{\partial k^*(A(0), s, \delta, n)}{\partial s} > 0, \\
\frac{\partial k^*(A(0), s, \delta, n)}{\partial n} &< 0 \text{ and } \frac{\partial k^*(A(0), s, \delta, n)}{\partial \delta} < 0,
\end{aligned}$$

and also

$$\begin{aligned}
\frac{\partial y^*(A(0), s, \delta, n)}{\partial A(0)} &> 0, \ \frac{\partial y^*(A(0), s, \delta, n)}{\partial s} > 0, \\
\frac{\partial y^*(A(0), s, \delta, n)}{\partial n} &< 0 \text{ and } \frac{\partial y^*(A(0), s, \delta, n)}{\partial \delta} < 0,
\end{aligned}$$

for each t.

Proposition 2.14. Suppose that Assumptions 1 and 2 hold, then the Solow growth model with Harrod-neutral technological progress and population growth in continuous time is asymptotically stable, i.e., starting from any $k(0) > 0$, the effective capital-labor ratio converges to a steady-state value k^* ($k(t) \to k^*$).

2 Equations

$$Y(t) = F[K(t), L(t), A(t)] \tag{2.1}$$

$$\lambda^m g(x, y, z) = g(\lambda x, \lambda y, z) \tag{2.2}$$

$$L(t) = \bar{L}(t) \tag{2.3}$$

$$\max_{L(t), K(t)} F[K(t), L(t), A(t)] - w(t) L(t) - R(t) K(t) \tag{2.4}$$

$$w(t) = F_L[K(t), L(t), A(t)] \tag{2.5}$$

$$R(t) = F_K[K(t), L(t), A(t)] \tag{2.6}$$

$$K(t+1) = (1-\delta) K(t) + I(t) \tag{2.7}$$

$$Y(t) = C(t) + I(t) \tag{2.8}$$

$$S(t) = I(t) = Y(t) - C(t) \tag{2.9}$$

$$C(t) = (1-s) Y(t) \tag{2.10}$$

$$K(t+1) = sF[K(t), L(t), A(t)] + (1-\delta) K(t) \tag{2.11}$$

$$k(t) \equiv \frac{K(t)}{L} \tag{2.12}$$

$$y(t) = F\left[\frac{K(t)}{L}, 1, A\right] \equiv f(k(t)) \tag{2.13}$$

$$\begin{aligned} R(t) &= f'(k(t)) > 0 \text{ and} \\ w(t) &= f(k(t)) - k(t) f'(k(t)) \end{aligned} \tag{2.14}$$

$$Y(t) = F[K(t), L(t)] = AK(t)^{\alpha} L(t)^{1-\alpha},\ 0 < \alpha < 1 \tag{2.15}$$

$$k(t+1) = sf(k(t)) + (1-\delta) k(t) \tag{2.16}$$

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s} \tag{2.17}$$

$$y^* = f(k^*) \tag{2.18}$$

$$c^* = (1-s) f(k^*) \tag{2.19}$$

$$\frac{\partial [f(k)/k]}{\partial k} = \frac{f'(k) k - f(k)}{k^2} = -\frac{w}{k^2} < 0 \tag{2.20}$$

$$\frac{\partial c^* (s)}{\partial s} = [f' (k (s)) - \delta] \frac{\partial k^*}{\partial s} \tag{2.21}$$

$$f' \left(k^*_{gold}\right) = \delta \tag{2.22}$$

$$\mathbf{x} (t + 1) = \mathbf{G} (\mathbf{x} (t)) \tag{2.23}$$

$$\mathbf{x} (t + 1) = \mathbf{A}\mathbf{x} (t) + \mathbf{b} \tag{2.24}$$

$$\mathbf{x} (t + 1) = \mathbf{G} [\mathbf{x} (t)] \tag{2.25}$$

$$k (t + 1) = g (k (t)) \tag{2.26}$$

$$k^* = g (k^*) \tag{2.27}$$

$$f (k) > f (0) + kf' (k) \geq kf' (k) \tag{2.28}$$

$$x (t + 1) - x (t) = g (x (t)) \tag{2.29}$$

$$\lim_{\Delta t \to 0} \frac{x (t + \Delta t) - x (t)}{\Delta t} = \dot{x} (t) \simeq g (x (t)) \tag{2.30}$$

$$L (t) = \exp (nt) L (0) \tag{2.31}$$

$$\frac{\dot{k} (t)}{k (t)} = s \frac{f (k (t))}{k (t)} - (n + \delta) \tag{2.32}$$

$$\frac{f (k^*)}{k^*} = \frac{n + \delta}{s} \tag{2.33}$$

$$\dot{\mathbf{x}} (t) = \mathbf{A}\mathbf{x} (t) + \mathbf{b} \tag{2.34}$$

$$\dot{\mathbf{x}} (t) = \mathbf{G} [\mathbf{x} (t)] \tag{2.35}$$

$$\sigma \equiv - \left[\frac{\partial \ln (F_K / F_L)}{\partial \ln (K/L)}\right]^{-1} \tag{2.36}$$

$$\begin{aligned} Y (t) &= F [K (t) , L (t) , \mathbf{A} (t)] \\ &\equiv A_H (t) \left[\gamma (A_K (t) K (t))^{\frac{\sigma - 1}{\sigma}} + (1 - \gamma) (A_L (t) L (t))^{\frac{\sigma - 1}{\sigma}}\right]^{\frac{\sigma}{\sigma - 1}} \end{aligned} \tag{2.37}$$

$$F [K (t) , L (t) , A (t)] = AK (t) \tag{2.38}$$

$$F [K (t) , L (t) , A (t)] = AK (t) + BL (t) \tag{2.39}$$

$$F [K (t) , L (t) , \mathbf{A} (t)] = A_H (t) \tilde{F} [A_K (t) K (t) , A_L (t) L (t)] \tag{2.40}$$

$$\frac{\dot{A}(t)}{A(t)} = g \tag{2.41}$$

$$\dot{K}(t) = sF\left[K(t), A(t)L(t)\right] - \delta K(t) \tag{2.42}$$

$$k(t) \equiv \frac{K(t)}{A(t)L(t)} \tag{2.43}$$

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - g - n \tag{2.44}$$

$$\frac{\dot{k}(t)}{k(t)} = \frac{sf(k(t))}{k(t)} - (\delta + g + n) \tag{2.45}$$

$$\frac{f(k^*)}{k^*} = \frac{\delta + g + n}{s} \tag{2.46}$$