Today: Primarily numerical analysis

$$s_{ij} = \frac{\exp\{X_{ij}'\beta\}}{\sum_{k=1}^{K}\exp\{X_{ik}'\beta\}}$$

- McFadden 1974
  - Random utility specification: $u_{ij} = X_{ij}'\beta + \varepsilon_{ij}$, $\varepsilon_{ij} \overset{iid}{\sim} EV1$ gives this form of mkt share

- Big assumption: β is constant across people
  - ignores consumer heterogeneity

We depart from this: assume $\beta \sim F(\beta)$
- we can estimate this distribution
- standard case: $\beta \sim N(b, \Sigma)$, $\beta = b + \Sigma^{1/2}u$, $u \sim N(0, I)$

We then look at:

$$E[s_{ij}] = \int_{\mathbb{R}^p} \frac{\exp\{X_{ij}'\beta\}}{\sum_{k=1}^{K}\exp\{X_{ik}'\beta\}} dF(\beta) \quad \text{where } \dim(\beta) = p$$

- This is sort of a nasty integral in general

<u>Monte-Carlo integration:</u>

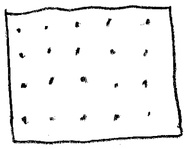
By the LLN,

$$E[s_{ij}] \approx \frac{1}{R}\sum_{r=1}^{R} \frac{\exp\{X_{ij}'\bar{\beta}_r\}}{\sum_{k=1}^{K}\exp\{X_{ik}'\bar{\beta}_r\}}, \text{ where } \bar{\beta}_r,\ r=1,\dots,R \text{ are draws from } F(\beta).$$

- Main problem: Need R very large (there is a curse of dimensionality built into p.)
  - clustering in random draws can produce junk
- There are problems estimating covariances
- This approximation has a non-negligible variance

Variance reduction techniques

- Halton sequences (used in an EMA 2006 paper)
  Instead of random draws, draw on a grid:



  - the draws are correlated

  - These originate from an obscure paper ('67)
  - The fact that these are correlated reduces variance without affecting the mean.
- How to construct these sequences? (see Train's book)
  - $S_0 = 0$
  - $S_{t+1} = \{ S_t, S_t + \frac{1}{k^t}, \ldots, S_t + \frac{(k+1)}{k^t} \}$
  - k prime
- Econometric Society website has some code for implementing this.
- This is a deterministic sequence. We make it random by adding a little bit of noise.
- Lattice Models - finds the optimal pattern for a particular integral. The optimal pattern does not look like a Halton sequence. What is the optimal pattern?

Importance Sampling

- Suppose we want to draw ε with pdf $f(\varepsilon)$, but it is difficult to do so
- Suppose $\exists$ a pdf g that is easier to

sample from (f, g must have same support)

- $\int \underbrace{t(\varepsilon)}_{\substack{\text{some statistic} \\ \text{(e.g. } s_{ij})}} f(\varepsilon)\,d\varepsilon = \int t(\varepsilon) \frac{f(\varepsilon)}{g(\varepsilon)} g(\varepsilon)\,d\varepsilon$
  - now, we can draw from g
- We can approximate this with:

$$\frac{1}{R} \sum_{r=1}^{R} t(\bar{\varepsilon}_r) \frac{f(\bar{\varepsilon}_r)}{g(\bar{\varepsilon}_r)} \equiv \hat{\theta}$$

- The variance is given by:

$$\mathrm{var}(\hat{\theta}) = \frac{1}{R} \int \left\{ \frac{f(\varepsilon)}{g(\varepsilon)} - \theta \right\}^2 g(\varepsilon)\,d\varepsilon$$

  - clearly, if we could pick $g = \frac{f}{\theta}$, this would minimize this variance, but we don't know θ.
  - want to choose something where $\frac{f}{g}$ is constant over the whole domain.

analytic approximations to the integral

$$E_\beta[s_{ij}] = E_\beta\left[\left(\sum_k \exp\{X_{ijk}'\beta\}\right)^{-1}\right], \quad X_{ijk} = X_{ik} - X_{ij}$$

with $\beta \sim N(b, \Sigma)$

$$\Rightarrow E_\beta[s_{ij}] = (2\pi)^{-p/2} |\Sigma|^{-1/2} \int \exp(-g(\beta))\,d\beta,$$

where $g(\beta) = \frac{1}{2}(\beta - b)'\Sigma^{-1}(\beta - b) + \log\left(\sum_{k=1}^{K} \exp\{X_{ijk}'\beta\}\right)$

Taylor Expansion

- expand around $\hat{\beta}_{ij}$ s.t. $g(\hat{\beta}_{ij}) < g(\beta)\ \forall \beta$
  - g is concave, so this minimum is well-defined
- $$g(\beta) = g(\hat{\beta}_{ij}) + (\beta - \hat{\beta}_{ij}) \frac{\partial g}{\partial \beta}\bigg|_{\beta = \hat{\beta}_{ij}} + \frac{1}{2}(\beta - \hat{\beta}_{ij})' \underbrace{\frac{\partial^2 g}{\partial \beta \partial \beta'}\bigg|_{\beta = \hat{\beta}_{ij}}}_{\equiv \tilde{\Sigma}_{ij}} (\beta - \hat{\beta}_{ij}) + \text{remainder}$$

Since $g(\hat{\beta}_{ij}) < g(\beta)\ \forall \beta$, $\frac{\partial g}{\partial \beta}\Big|_{\beta = \hat{\beta}_{ij}} = 0$

$$\Rightarrow E_\beta[s_{ij}] \approx |\Sigma|^{-1/2} \exp\{-g(\hat{\beta}_{ij})\} |\tilde{\Sigma}_{ij}|^{1/2} \underbrace{(2\pi)^{-p/2} |\tilde{\Sigma}_{ij}|^{-1/2} \int \exp\left\{-\frac{1}{2}(\beta - \hat{\beta}_{ij}) \tilde{\Sigma}_{ij}^{-1} \cdot (\beta - \hat{\beta}_{ij})\right\} d\beta}_{=1}$$

$$\Rightarrow E_\beta[s_{ij}] \approx \sqrt{\frac{|\tilde{\Sigma}_{ij}|}{|\Sigma|}} \exp\{-g(\underbrace{\hat{\beta}_{ij}}_{\text{solution to a well-behaved fixed point}})\}$$

- once you have $\hat{\beta}_{ij}$, you can derive $\tilde{\Sigma}_{ij}$

This approximation is good for small mkt shares
- Makes it very easy to estimate covariances.
  - important for counterfactuals

The other paper deals with much more general integrals.