Last time:

$B_i^p(E) = \{\omega \mid \mu(E \mid I_i(\omega)) \geq p\}$

E is p-evident if $E \subseteq B_i^p(E)$ $\forall i$.

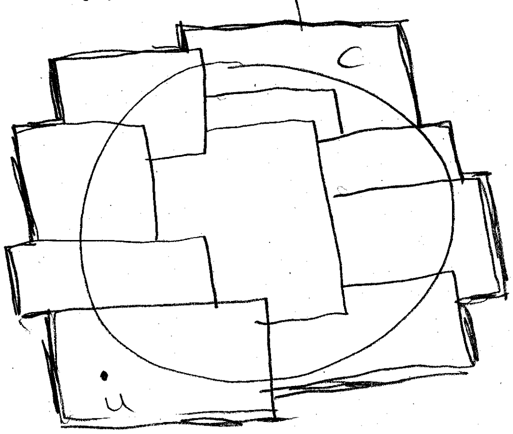
C is a common p-belief at ω if $\exists\, E \ni \omega$ s.t.

- $E \subseteq B_i^p(E)$ $\forall i$
- $E \subseteq B_i^p(C)$ $\forall i$

$$E^p(C) = \left(\bigcap_{i=1}^{n} B_i^p(C)\right) \cap \left(\bigcap_{i=1}^{n} B_i^p\left(\bigcap_{i=1}^{n} B_i^p(C)\right)\right) \cap \ldots$$

Generalizing the "agreeing to Disagree" result.

Thm: Let X be an event, and define $f_i(\omega) = \mu(X \mid I_i(\omega))$. Consider a sequence $(r_1, r_2, \ldots, r_n)$ s.t. $r_i \in [0,1]$.
Let $C = \{u \mid f_i(u) = r_i \ \forall i\}$. Then, if C is a
not utility
common p-belief at ω, $|r_i - r_j| \leq 2(1-p)$ $\forall i,j$.



- $f_i(u) = r_i$
- $f_i(u) = r_i$ for every point in this region.

Pf:

1] $f_i(u) = r_i \quad \forall u \in S \in I_i$ s.t. $S \cap C \neq \emptyset$.

$\Rightarrow f_i(u) = r_i \quad \forall u \in B_i^p(C)$ since if $I_i(u) \cap C \neq \emptyset$, $u \in B_i^p(C)$

2] Suppose C is a common p-belief at ω. This implies that $\exists \; E \ni \omega$ s.t. $E \subseteq B_i^p(E) \; \forall i$ and $E \subseteq B_i^p(C)$

3] By 1], $f_i = r_i$ on E. Indeed $f_i = r_i$ on $B_i^p(E)$.

4] $\mu(X \mid B_i^p(E)) = r_i$

Main step: Let $x = \mu(X \mid E) = \dfrac{\mu(X \cap E)}{\mu(E)}$

$$= \underbrace{\frac{\mu(X \cap B_i^p(E))}{\mu(E)}} - \frac{\mu(X \cap [B_i^p(E) \setminus E])}{\mu(E)}$$

$$= \frac{\mu(X \cap B_i^p(E))\, \mu(B_i^p(E))}{\mu(B_i^p(E))\, \mu(E)}$$

$$= \underbrace{\mu(X \mid B_i^p(E))}_{= r_i} \cdot \frac{\mu(B_i^p(E))}{\mu(E)}$$

$$\Rightarrow r_i = x \cdot \frac{\mu(E)}{\mu(B_i^p(E))} + \frac{\mu(X \cap (B_i^p(E) \setminus E))}{\mu(B_i^p(E))}$$

o want to find bounds on this.

$$\mu(B_i^p(E)) \geq \mu(E) \geq p\,\mu(B_i^p(E))$$

since $\forall u \in \mu(B_i^p(E))$, $\mu(E \cap I_i(u)) \geq p\mu(I_i(u))$

$$\Rightarrow \underbrace{\sum \mu(E \cap I_i(\omega))}_{\mu(E)} \geq p \underbrace{\sum \mu(I_i(u))}_{\mu(B_i^p(E))}$$

Thus, $$1 \geq \frac{\mu(E)}{\mu(B_i^p(E))} \geq p$$

So that

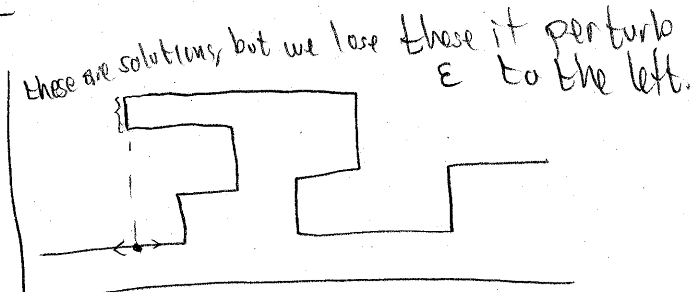
$$r_i = x_i \underbrace{\frac{\mu(E)}{\mu(B_i^p(E))}}_{\in [p,1]} + \underbrace{\frac{\mu(X \cap (B_i^p(E) \setminus E))}{\mu_i(B_i^p(E))}}_{\in [0,1-p]}$$

$$\Rightarrow \quad p x \leq r_i \leq x + 1 - p$$

$$\Rightarrow \quad -(1-p)x \leq r_i - x \leq 1-p$$

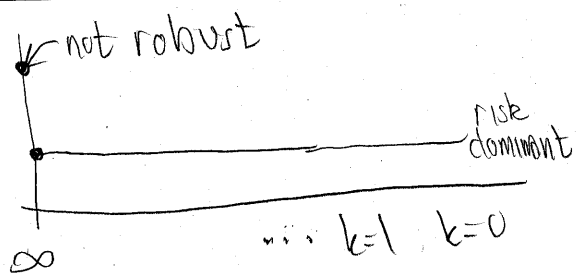
$$\Rightarrow \quad |r_i - x| \leq 1-p$$

$$\Rightarrow \quad |r_i - r_j| \leq |r_i - x| + |r_j - x| \leq 2(1-p) \quad \square$$



uhc - cannot gain additional solutions by moving ε to any side

lhc - cannot lose a solution by moving ε away (i.e. robust.)



Rubinstein's e-mail game.

- want to define a new topology s.t. this correspondence is lhc wrt. this topology.

- Want to relax common knowledge to common p-belief, where p is high.

Common p-belief in Bayesian Games

Let $G^1, \ldots, G^j, \ldots, G^m$ be games with

$G^j = (N, \underbrace{A_i}_{\text{finite}}, \underbrace{H_i^j}_{\text{utility functions}})$. Let $J: \underbrace{\Omega}_{\substack{\text{not necessarily finite} \\ \text{(could be countable)}}} \to \{1, \ldots, m\}$

s.t. $u_i(a, \omega) = H_i^{J(\omega)}(a)$.

a mixed strategy is a fcn $\sigma_i : \Omega \to \Delta(A_i)$

$F_i(\sigma) = E[H_i^J(\sigma)]$.

σ is an ε-equilibrium iff $\forall i, \omega, s_i$, we have:

$$E[H_i^J(\sigma) \mid I_i(\omega)] \geq E[H_i^J(s_i, \sigma_{-i}) \mid I_i(\omega)] - \varepsilon$$

Define $\mathcal{G}^j = \{\omega \mid J(\omega) = j\}$. Let $s^j \in NE(G^j)$. Consider σ^* s.t. $\sigma^*(\omega) = s^{J(\omega)}$.

Thm: Let $p \geq \frac{1}{2}$. Define $1-\delta \equiv \mu(\{\omega | G^j$ is common p-belief at $\omega\})$. Let $M = \max_{i,j,s} |H_i^j(s)|$. Then, $\exists\ \Omega'$ and σ s.t.,

$\boxed{1}$ $\mu(\Omega') \geq (2p-1)(1-\delta)$

$\boxed{2}$ $\sigma(\omega) = s^{J(\omega)}$ on Ω'

$\boxed{3}$ σ is $\underbrace{2M(1-(2p-1))}_{\varepsilon(p) = 4M(1-p)}$ equilibrium

- $\varepsilon(p) \to 0$ as $p \to 1$

For every original equilibrium, if the probability of this game being common p-belief is sufficiently high, then this equilibrium is an $\varepsilon(p)$ equilibrium of nearby games.

Pf: Define $E^j = \{\omega | G^j$ is common p-belief at $\omega\}$
$= E^p(G^j)$

Define $\Omega' = \bigcup_j (E^j \cap G^j)$. Define

$\Omega_i = \bigcup_j B_i^p(E^j)$ since $p > \frac{1}{2}$, $B_i^p(E^j)$, j are disjoint

$\forall\ \omega \in B_i^p(E^j)$, $\sigma_i(\omega) = s_i^j$

Consider the following game:

- players are i s.t. $\Omega_i^c \neq \emptyset$

- $s_i : \Omega_i^c \to A_i$
- $u_i(s) = H_i(s, \sigma)$
- $(s,\sigma)_i(\omega) = \begin{cases} \sigma_i(\omega) & \text{if } \omega \in \Omega_i \\ s_i & \text{otherwise} \end{cases}$

Let $\hat{\sigma}$ be a Bayesian Nash Equilibrium of this game.

Then $\forall\, \omega \in \Omega_i^c$, $\sigma_i(\omega) = \hat{\sigma}_i(\omega)$.

First, to prove 2, if $\omega \in E^j \subseteq B_i^p(E^j)$, then we have that $\sigma_i(\omega) = s_i^{J(\omega)}$.

Next for 1, recall $E^j \subseteq B_i^p(E^j)$ and $E^j \subseteq B_i^p(G^j)$. These imply $E^j \subseteq B_i^{2p-1}(E^j \cap G^j)$

- $\underbrace{\mu_{i,\omega}(E^j) + \mu_{i,\omega}(G^j) - \mu_{i,\omega}(E^j \cap G^j)}_{\mu_{i,\omega}(E^j \cup G^j)} \leq 1$

  $\geq p \qquad \geq p \qquad \Rightarrow \geq 2p-1$

- $\mu(E^j \cap G^j \mid B_i^{2p-1}(E^j \cap G^j)) \geq 2p-1$

Then, we know that $\mu(E^j \cap G^j) \geq (2p-1)\,\mu(B_i^{2p-1}(E^j \cap G^j))$

$\geq (2p-1)\,\mu(E^j)$

Take the sum over j: $\mu(\Omega') \geq (2p-1)(1-\delta)$.

Finally, to show 3, when $\omega \in \Omega_i^c$, you are playing a best reply. Consider $\omega \in \Omega_i$.

- Take $\omega \in B_i^p(E^j)$. Then $\mu(E^j \cap G^j \mid I_i(\omega)) \geq 2p-1$. Then s_i^J is a best reply with probability greater than $2p-1$. The loss from s_i^J with respect to s_i' is less than or equal to $2M$ with probability $1-(2p-1)$. Thus, the loss is less than $2M(1-(2p-1))$. $\square$

Let $\varepsilon > 0$ and $M > 0$. There exists $p^0 < 1$, $\delta^0 > 0$ s.t. $\forall p > p^0$ and $\delta < \delta^0$, $\forall \{G^j\}$ with $\max_{i,j,s} |H_i^j(s)| \leq M$, $\forall$ equilibrium selection (s^j) for G^j's, and $\forall$ incomplete information games, if $\mu(\{\omega \mid \exists j \text{ s.t. } G^j \text{ is common p-belief}\}) \geq 1-\delta$, then there exists an ε-equilibrium σ s.t. $\mu(\{\sigma(\omega) = s^J(\omega)\}) \geq 1-\varepsilon$.