Interim rationalizability

Redundant types might play a significant role in strategic behavior.

Interim correlated rationalizability $\Leftrightarrow$ common knowledge of rationality

- upper-hemicontinuity

a game with CPA: $(N, A, u, \Theta \times T_1 \times T_2, p)$

- Payoffs:

θ	L	R
U	1,0	0,0
D	.6,0	.6,0

θ'	L	R
U	0,0	1,0
D	.6,0	.6,0

- Types:

θ	t_2'	t_2''
t_1'	1/6	1/12
t_1''	1/12	1/6

θ'	t_2'	t_2''
t_1'	1/12	1/6
t_1''	1/6	1/12

BNE: $s_1(t_1') = U$; $s_1(t_1'') = D$; $s_2(t_2') = L$; $s_2(t_2'') = R$

t_1' beliefs

	t_2'	t_2''
θ	1/3	1/6
θ'	1/6	1/3

t_1'' beliefs

	t_2'	t_2''
θ	1/6	1/3
θ'	1/3	1/6

$E[u_1(U, s_2) | t_1'] = \frac{1}{3}(1) + \frac{1}{6}(0) + \frac{1}{6}(0) + \frac{1}{3}(1) = \frac{2}{3}$

$E[u_1(D, s_2) | t_1'] = 0.6$

$2/3 > 0.6 \Rightarrow$ not want to deviate

$E[u_1(D, s_2) | t_1''] = 0.6$

$E[u_1(U, s_2) | t_1''] = \frac{1}{6}(1) + \frac{1}{3}(0) + \frac{1}{3}(0) + \frac{1}{6}(1) = \frac{1}{3}$

$1/3 < 0.6 \Rightarrow$ not want to deviate.

Thus, neither type of player 1 wants to deviate.
Player 2 is indifferent among everything, so he is playing a BR

another:
$s_1(t_1') = s_1(t_1'') = D,\ s_2(t_2') = s_2(t_2'') = L$

$$\left.\begin{aligned} Pr[\theta|t_1'] &= \tfrac{1}{2} = Pr[\theta'|t_1'] \\ Pr[\theta|t_1''] &= \tfrac{1}{2} = Pr[\theta'|t_1''] \\ Pr[\theta|t_2'] &= \tfrac{1}{2} = Pr[\theta'|t_2'] \\ Pr[\theta|t_2''] &= \tfrac{1}{2} = Pr[\theta'|t_2''] \end{aligned}\right\} \text{First-order beliefs}$$

$$Pr[(\theta, h_2^1(t_2))|t_1'] = \begin{cases} \frac{1}{2} & \text{if } h_2^1(t_2) = \frac{1}{2}\delta_\theta + \frac{1}{2}\delta_{\theta'} \\ 0 & \text{otherwise} \end{cases}$$

$$Pr[(\theta', h_2^1(t_2))|t_2'] = \begin{cases} \frac{1}{2} & \text{if } h_2^1(t_2) = \frac{1}{2}\delta_\theta + \frac{1}{2}\delta_{\theta'} \\ 0 \end{cases}$$

Common knowledge: $h_i^1(t_i) = \frac{1}{2}\delta_\theta + \frac{1}{2}\delta_{\theta'}$

Redundant types: $h_1(t_1') = h_1(t_1'')$ and $h_2(t_2') = h_2(t_2'')$

an "equivalent game": $(N, A, u, \Theta \times \{t_1\} \times \{t_2\}, p)$, where
$p(\theta) = p(\theta') = \frac{1}{2}$
- ie has same image in universal type space.

Ex ante:

θ

1,0	0,0
.6,0	.6,0

θ'

0,0	1,0
.6,0	.6,0

$\Rightarrow$

.5,0	.5,0
.6,0	.6,0

BNE is $s_1(t_1) = D_1$, $s_2(t_2) =$ anything

Knowing hierarchies of beliefs is not enough to determine the strategies people will play.

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- Need to consider a space larger than the universal type space.

Dekel, Fudenberg, Morris

- type spaces can introduce additional correlations not visible in belief hierarchies.

For each type t_i,

- $S_i^0[t_i; T] = A_i$
- $a_i \in S_i^k[t_i; T]$ iff $a_i \in \underset{a_i}{\operatorname{argmax}} \sum_{(\theta, t_{-i}, a_{-i})} u_i(\theta, a_i', a_{-i}) \mu(\theta, t_{-i}, a_{-i})$

  for some μ s.t.

  (1) $\mu(\theta, t_{-i}, a_{-i}) > 0 \Rightarrow a_{-i} \in S_{-i}^{k-1}[t_{-i}; T]$

  (2) $p_i(\theta, t_{-i} | t_i) = \sum_{a_{-i}} \mu(\theta, t_{-i}, a_{-i})$

Iteratively eliminate all actions which are <u>strictly dominated actions</u>.

$$S_i^\infty[t_i; T] = \bigcap_{k=0}^{\infty} S_i^k[t_i; T]$$

Interim Independent Rationalizability

- For each t_i

$$IS_i^0[t_i;T] = A_i$$

$a_i \in IS_i^k[t_i;T]$ iff

$$a_i \in \underset{a_i'}{\operatorname{argmax}}\, E[u_i(a_i', \sigma_{-i}(t_{-i}), \theta) \mid t_i]$$

for some $\sigma_j : t_j \mapsto \sigma_j(t_j) \in \Delta(IS_i^{k-1}[t_j;T])$, $j \neq i$

$$IS_i^\infty[t_i;T] = \bigcap_{k=0}^{\infty} IS_i^k[t_i;T]$$

$IS_i^k \subseteq S_i^k \quad \forall k \Rightarrow IS_i^\infty \subseteq S_i^\infty$

Thm: $\forall$ two models $(N, A, u, \Theta \times T, p)$ and $(N, A, u, \Theta \times T', p')$, $\forall\, t_i \in T_i$ and $t_i' \in T_i'$,

$$h_i^k(t_i) = h_i^k(t_i') \Rightarrow S_i^k[t_i;T] = S_i^k[t_i';T]$$

Pf: (By induction on k): Take $T' = T^*$. If the claim holds in the universal type space, it holds everywhere.

Let $a_i \in S_i^k[t_i;T]$. Then,

$a_i \in \underset{a_i'}{\operatorname{argmax}}\, E_\mu[u_i]$ for some μ s.t.

1] $\operatorname{marg}_{\Theta \times T_{-i}} \mu = p_i(\cdot \mid t_i)$

2] $\mu(\theta, t_{-i}, a_{-i}) > 0 \Rightarrow a_{-i} \in S_{-i}^{k-1}[t_{-i}; T]$

Want to find $\tilde{\mu} \in \Delta(\Theta \times T_{-i}^* \times A_{-i})$

Define: $\gamma_i : \Theta \times T_{-i} \times A_{-i} \to \Theta \times T_{-i}^* \times A_{-i}$, where

$\gamma(\theta, t_{-i}, a_{-i}) = (\theta, h_{-i}(t_{-i}), a_{-i})$

Define $\tilde{\mu} = \mu \circ \gamma^{-1}$. Want to show (0), (1), (2)

(0) $a_i \in \operatorname{argmax}_{a_i'} E_{\tilde{\mu}}[u_i]$

$\operatorname{marg}_{\Theta \times A_{-i}} \tilde{\mu} = \operatorname{marg}_{\Theta \times A_{-i}} \mu$, so (0) holds since

$a_i \in \operatorname{argmax}_{a_i'} E_{\mu}[u_i]$

(1) $\operatorname{marg}_{\Theta \times T_{-i}^*} \tilde{\mu} = \kappa_{h_i(t_i)}$. This holds by the definition of γ

(2) $\tilde{\mu}(\theta, h_{-i}(t_{-i}), a_{-i}) > 0 \Rightarrow \mu(\theta, t_{-i}, a_{-i}) > 0$

$\Rightarrow a_{-i} \in S_{-i}^{k-1}[t_{-i}; T] = S_{-i}^{k-1}[h_{-i}(t_{-i}); T]$,

which holds by induction. $\square$

Trivially holds for $k=1$, since $S_{-i}^1[t_{-i}; T] = S_{-i}^1[h_{-i}(t_{-i}); T] = A_{-i}$

Fixed point definition

S^∞ is the largest set $Z = Z_1 \times \ldots \times Z_n \subseteq AT$ such that $\forall\, a_i \in Z_i(t_i)$, $Z_i : T_i \to 2^{A_i}$

$$a_i \in \underset{a_i'}{\operatorname{argmax}} \sum_{(\theta, t_{-i}, a_{-i})} u_i(\theta, a_i', a_{-i})\, \mu(\theta, t_{-i}, a_{-i}) \text{ for some } \mu \text{ s.t.}$$

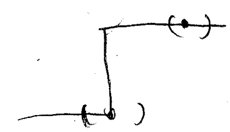
1] $\mu(\theta, t_{-i}, a_{-i}) > 0 \Rightarrow a_{-i} \in Z_{-i}[t_{-i}]$

2] $p_i(\theta, t_{-i} \mid t_i) = \sum_{a_{-i}} \mu(\theta, t_{-i}, a_{-i})$

Upper-hemicontinuity

$S^k : T^* \to 2^A$ is uhc: for each t_i, $\exists$ a nbhd $\mathcal{V}$ s.t.

$$S^k[t'] \subseteq S^k[t] \quad \forall\, t' \in \mathcal{V}$$



Consider $U = \{t \mid |S^\infty[t]| = 1\}$

- $\forall\, t \in U$, $\exists\, \mathcal{V}(t)$ s.t. $S^\infty[t'] \subseteq S^\infty[t]$ $\forall\, t' \in \mathcal{V}(t)$
  $\neq \emptyset$
  $\Rightarrow S^\infty[t'] = S^\infty[t]$
  $\Rightarrow U$ is open.

$f : U \to A$ continuous $\Rightarrow S^\infty[t] = \{f(t)\}$

Upper-hemicontinuity is considered to be a very desirable property for any solution concept.
Interim independent rationalizability is <u>not</u> uhc.