Epistemic Foundations of Solution Concepts

- Modeling strategic uncertainty
- Correlated equilibrium (Rationalizability + common prior about actions)
- Rationalizability (eq. to common knowledge about rationalizability)
- Nash Equilibrium (doesn't require strong epistemic assumptions)

Epistemic Model: List $(N, A, u, I, \Omega, p, s)$

- $N = \{1, \ldots, n\}$ players
- $A = A_1 \times \ldots \times A_n$ actions
- $u_i : A \to \mathbb{R}$ payoffs
- Ω - state space
- I_i defines an information partition for i
- For each i, ω, $p_{i\omega} \in \Delta(I_i(\omega))$ belief
- $s : \Omega \to A$ strategy function; σ_i constant over $I_i(\omega)$

Notes and definitions:

- To add uncertainty, let $u_i : A \times \Omega \to \mathbb{R}$
- only one ω represents the actual world
- s gives strategic meaning to states
- conjecture of i about others' strategies:
  - $\varphi^i_\omega = p_{i\omega} \circ s_{-i}^{-1}$

$I_1(\omega)$

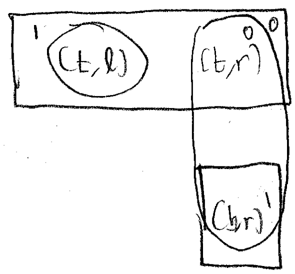
[1/3]	[1/3]	[1/3]
(a,b)	(a,c)	(a,c)

- 1 believes
  - b played w/ prob 1/3
  - c played w/ prob 2/3

- common prior assumption:
  - $p_{i,\omega} = p(\cdot \mid I_i(\omega)) \qquad (\forall i, \omega)$

- differences in beliefs represent differences in information only. p is not indexed

	l	r
t	2,1	0,0
b	0,0	1,2

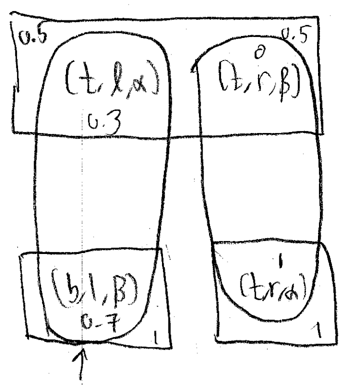


α

2,1	0,0
0,0	1,2

β

2,2	0,3
3,0	1,1



here, 1 is rational, 2 is not rational,
1 knows 2 is not rational

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	c	d
C	2,2	0,0
D	0,0	1,1

	d	d2	d3
C1	.5,.5	.5,.5	0,0
D1	.5,.5	0,0	.5,.5
D2	0,0	.5,.5	.5,.5

- playing D,d is mutually known
- it is not known up to higher order

Hierarchies of Beliefs

- First-order beliefs : b_i^1 = probability distribution on a_{-i}
- Second-order beliefs : b_i^2 = probability distribution on (a_{-i}, b_{-i}^1)
  - depends on what you play and what you think I am doing.
- Third-order beliefs: b_i^3 = prob. distr. on $(a_i, b_{-i}^1, b_{-i}^2)$

- Hierarchies of beliefs: $b_i = (b_i^1, b_i^2, b_i^3, \dots)$
  - this is a difficult object to work with
- coherency: (*) working with incoherent beliefs could be interesting
  - $\operatorname{marg}_{s_{-i}, b_{-i}^1, \dots, b_{-i}^{k-2}} b_i^k = b_i^{k-1}$
  - Can integrate out higher-order beliefs to acquire lower-order beliefs

Alternative formulation

- $X_0 = A \Rightarrow \Delta(X_0)$ - space of first-order beliefs
- $X_1 = X_0 \times (\Delta(X_0))^n \Rightarrow \Delta(X_1)$ - space of second-order beliefs
- $X_2 = X_1 \times (\Delta(X_1))^n$
- $X_m = X_{m-1} \times (\Delta(X_{m-1}))^n \Rightarrow \Delta(X_m)$ - space of m^{th}-order beliefs
- $\Delta(X_0) \times \Delta(X_1) \times \Delta(X_2) \times \dots$ - hierarchy of beliefs
- coherence: players know their own beliefs and b_i^k contains info. about b_i^{k-1}; $\operatorname{marg}_{X_{k-2}} b_i^k = b_i^{k-1}$

a player is rational at ω if ω maximizes, s.t. beliefs, his expected utility: $s_i(\omega) \in BR_i(\varphi^i_\omega)$

i.e. if Ω finite, $\forall a_i \in A_i$,

$$\sum_{\omega' \in I_i(\omega)} u_i(s_i(\omega), s_{-i}(\omega')) p_{i,\omega}(\omega') \geq \sum_{\omega' \in I_i(\omega)} u_i(a_i, s_{-i}(\omega')) p_{i,\omega}(\omega')$$

- common knowledge of rationality says that everyone plays a BR everywhere.
  - $CK(\underbrace{R}_{\text{rationality}}) \Rightarrow s: \Omega \to A$ is a Nash Equilibrium
  - $R = \{\omega \mid \text{each } i \text{ is rational at } \omega\} \subseteq \Omega$

Correlated Equilibrium

- Given a finite strategic form game (N, A, u), a <u>correlated strategy profile</u> consists of:
  - A finite probability space $\overbrace{(\Omega', \pi)}^{\text{describes the randomizing device}}$
  - For each i, an information partition P_i of Ω' (e.g. Bayesian posterior)
  - For each i: $\sigma_i : \Omega' \to A_i$ measurable wrt P_i.
- a corr. strat. prof σ is a correlated equilibrium if $\forall i$ and for each τ_i measurable wrt P_i,

$$\sum_{\omega \in \Omega'} u_i(\sigma(\omega))\pi(\omega) \geq \sum_{\omega \in \Omega'} u_i(\tau_i(\omega), \sigma_{-i}(\omega))\pi(\omega)$$

CPA + CK(Rationality) = CE

<u>Thm:</u> Let $(N, A, u, \Omega, I, p, s)$ be a model w/ finite Ω. Assume CPA. If each i is rational at each ω, then the distribution of s_i is a correlated equilibrium distribution.

<u>Pf:</u> Define $(\Omega', \mathcal{P}_i, \pi, \sigma_i) \equiv (\Omega, I, p, s)$. Fix i, $\mathcal{P}_i$-measurable τ_i. For any ω, by rationality,

$$\sum_{\omega' \in I(\omega)} u_i(s_i(\omega), s_{-i}(\omega')) \underbrace{p(\omega' | I_i(\omega))}_{\equiv p_{i,\omega}(\omega')} \geq \sum_{\omega' \in I_i(\omega)} u_i(\tau_i(\omega), s_{-i}(\omega')) p(\omega' | I_i(\omega))$$

summing up over $I_i(\omega)$: (weighted by $p(I_i(\omega))$)

$$\sum_{\omega' \in \Omega} u_i(s_i(\omega), s_{-i}(\omega')) p(\omega') \geq \sum_{\omega' \in \Omega} u_i(\tau_i(\omega), s_{-i}(\omega')) p(\omega')$$

<u>Rationalizability</u>

- $S_0 = A$
- $S_i^k = BR_i(\Delta(S_{-i}^{k-1}))$

  $\vdots$
- $S_i^\infty = \bigcap_{k=0}^{\infty} S_i^k$ — (correlated) rationalizable strategies

<u>Thm:</u> (Fixed pt definition): S^∞ is the largest set $Z_1 \times \ldots \times Z_n$ s.t. $Z_i \subseteq BR_i(\Delta(Z_{-i}))\ \forall i$. ($a_i$ is rationalizable iff $a_i \in Z_i$ for such $Z_1 \times \ldots \times Z_n$)

Thm: If each i is rational at each ω, then $s(\Omega) \subseteq S^\infty$. There is a model with $s(\Omega) = S^\infty$ and CKR. (ie cannot make any progress w/out assuming more than CKR.)

Pf: For each $a_i = s_i(\omega) \in s_i(\Omega)$, by CKR,

- $a_i \in BR_i(p_{i,\omega} \circ s_{-i}^{-1}) \subseteq BR_i(\Delta(s_{-i}(\Omega)))$

$\Rightarrow s(\Omega) \subseteq S^\infty$.

For the second part, take

$\Omega \Leftarrow$ - $\Omega = S^\infty$

$I \Leftarrow$ - $I_i(a) = \{a' \mid a_i' = a_i\}$ (ie they know their own strategies)

$p \Leftarrow$ - $a_i \in BR_i(p_{i,a})$ - define $p_{i,a}$ implicitly here

$s \Leftarrow$ - $s_i(a) = a_i$

Thus, we have $(N, A, u, \Omega, I, p, s)$ s.t. $s(\Omega) = S^\infty$.