Recall the monotonicity theorem:
Let $f: X \times \mathbb{R} \to \mathbb{R}$ be supermodular and define

$$x^*(t) = \underset{x \in S(t)}{\operatorname{argmax}}\ f(x,t)$$

If $t \geq t'$ and $S(t) \geq S(t')$, then $x^*(t) \geq x^*(t')$.

Monopolist

- $D(p,t)$ demand
- c - cost

- When is p^* increasing in c?
- When is p^* increasing in t?

$$p^*(c,t) = \underset{p}{\operatorname{argmax}}\ (p-c)\,D(p,t)$$

$$= \underset{p}{\operatorname{argmax}}\ \underbrace{\log(p-c) + \log D(p,t)}_{f(p,c,t)}$$

$$\frac{\partial f}{\partial c} = \frac{\partial \log(p-c)}{\partial c} = -\frac{1}{p-c}$$ is increasing in p

$$\frac{\partial^2 f}{\partial p \partial c} = \frac{1}{(p-c)^2} \geq 0$$ supermodular in (p,c)

- will have $p^*(c,t)$ isotone in t if

$$\frac{\partial \varepsilon_{D,p}}{\partial t} \geq 0$$

Auction

- First price auction

- Bidder gets $u(p,t)$ if win at price p
  - t is bidder's type.
  - p wins with probability $F(p)$.

Is $p(t)$ nondecreasing in t for any function F?

$$p^*_B(t) = \arg\max_p u(p,t)F(p)$$

$$= \arg\max_p \log u(p,t) + \log F(p)$$

$p^*_B(t)$ is nondecreasing in t if $\log u(p,t)$ is supermodular in (p,t)

<u>Producer theory</u>

$$\max_{l,k} \overbrace{pf(k,l) - L(l,w) - K(k,r)}^{=F(k,l,w,r)}$$

- suppose L is supermodular in (l,w)
  - then $-L$ is supermodular in $(l,-w)$

$\Rightarrow l^*(w)$ is nondecreasing in $-w$

- ie $l^*(w)$ is nonincreasing in w.

Cannot get this directly from the theorem, since we would need to impose some restrictions on f.

Need that all cross partials are non-negative in some order.

- use the fact that $F_{kw} = F_{lr} = 0$ and change the ordering on k, l to get the result.

If f is supermodular, then $k^*(w)$ is also nonincreasing.

- capital and labor are complements if f is supermodular.

If f is submodular, then k and l are substitutes, since $k^*(w)$ is nondecreasing.

Need more than just a single cross-partial.

Topology of Lattices

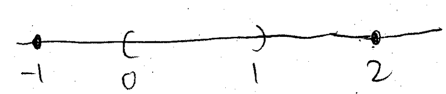
Let $(X, \geq)$ be a complete lattice

Let τ be a topology such that

for any $\{x_m\}_{m>0}$ with $x_m \geq x_{m+1}$ $\forall m$, $x_m \to \inf\{x_m, m>0\}$

for any $\{x_m\}_{m>0}$ with $x_{m+1} \geq x_m$ $\forall m$, $x_m \to \sup\{x_m, m>0\}$

e.g. $X = \{-1\} \cup (0, 1) \cup \{2\}$



in this topology, $x_m = \frac{1}{m} \Rightarrow x_m \to -1$

Let $U(x) = \begin{cases} x & x \in (0,1) \\ 1 & x = 2 \\ 0 & x = -1 \end{cases}$

- This is continuous with respect to our topology.

Supermodular games

- N players - can be infinite
- Strategy sets $(X_n, \geq_n)$ complete lattice
  - Let $\underline{x}_n = \min\{X_n\}$, $\bar{x}_n = \max\{X_n\}$
- Payoff functions $U(x)$ are continuous.
  - If $x^m \to x$, $U(x^m) \to U(x)$
- Supermodular in own strategies and has increasing differences with others' strategies
  - i.e. $\forall n$, $\forall x_n, x_n' \in X_n$ $\forall x_{-n} \geq x_{-n}' \in X_{-n}$,

$$U_n(x) + U_n(x') \leq U_n(x \wedge x') + U_n(x \vee x')$$

- Such a game is referred to as a supermodular game.

Bertrand

- $Q_n(x) = A - a x_n + \sum_{j \neq n} b_j x_j$

- Profit: $U_n(x) = (x_n - c_n)\, Q_n(x)$

$$\frac{\partial U_n}{\partial x_m} = b_m(x_n - c_n) \Rightarrow \frac{\partial^2 U_n}{\partial x_m \partial x_n} = b_m \geq 0$$

Cournot:

- $P(x) = A - x_1 - x_2$
- $U_n(x) = x_n P(x) - C_n(x_n)$
- $\frac{\partial U_n}{\partial x_m} = -x_n \Rightarrow \frac{\partial^2 U_n}{\partial x_m \partial x_n} = -1 \leq 0 \Rightarrow$ submodular u-fcn
  - can put reverse order if there are exactly two players. cannot do the trick to obtain supermodularity with more than two players.

(*) Expectation operator preserves supermodular.

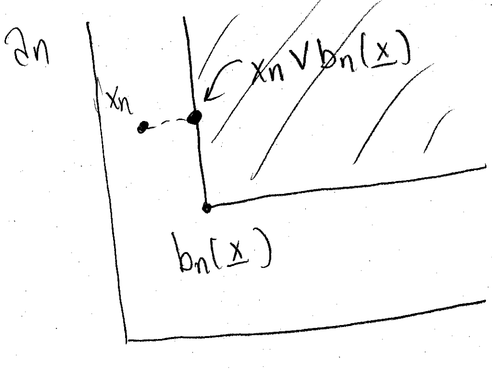
Best reply functions:

$$B_n(x) = \max\{\operatorname*{argmax}_{x_n' \in X_n} U_n(x_n', x_{-n})\}$$

$$b_n(x) = \min\{\operatorname*{argmax}_{x_n' \in X_n} U_n(x_n', x_{-n})\}$$

- By Topkis' theorem, $B_n(x) \geq B_n(x')$ if $x_{-n} \geq x_{-n}'$ and $b_n(x) \geq b_n(x')$ if $x_{-n} \geq x_{-n}'$

Lemma: Suppose $x_n \not\geq b_n(\underline{x})$. Then x_n is strictly dominated by $\underbrace{b_n(\underline{x})}_{\text{best reply to other players' minimal strategy}} \vee x_n$.



a_n - advertising
q_n - quantity

Pf: $\underbrace{U_n(x_n \vee b_n(\underline{x}), x_{-n}) - U_n(x_n, x_{-n})}_{\text{want to show that this is } > 0}$ (1)

Compare it to:

$U_n(b_n(\underline{x}), \underline{x}_{-n}) - U_n(x_n \wedge b_n(\underline{x}), \underline{x}_{-n})$

By supermodularity in own payoffs

$$(1) \geq U_n(b_n(\underline{x}), x_{-n}) - U_n(x_n \wedge b_n(\underline{x}), x_{-n})$$

$$\geq U_n(b_n(\underline{x}), \underline{x}_{-n}) - U_n(x_n \wedge b_n(\underline{x}), \underline{x}_{-n})$$

by increasing differences

Since $x_n \not\geq b_n(\underline{x})$, $U_n(b_n(\underline{x}), \underline{x}_{-n}) - U_n(x_n \wedge b_n(\underline{x}), \underline{x}_{-n}) > 0$

or else $x_n \wedge b_n(\underline{x})$ is a best reply.

$\Rightarrow x_n \geq b_n(\underline{x}) \rightarrow\leftarrow$.