I Supermodularity vs. single crossing

II Some facts about lattices and supermodular functions

III Milgrom Roberts (1990)

IV adapting this logic

I Single crossing

- complementarities across actions
- $x^*(t) = \arg\max_{x \in S(t)} f(x,t)$, $t \geq t'$, $S(t) \geq S(t') \Rightarrow x^*(t) \geq x^*(t')$

  if f is supermodular

Note: f supermodular $\not\Rightarrow$ $g(f)$, g increasing is supermodular, even though g is strictly increasing.

$$\frac{\partial^2}{\partial x \partial y} g(f(x,y)) = \underbrace{g'}_{\geq 0} \underbrace{\frac{\partial^2 f}{\partial x \partial y}}_{\geq 0} + \underbrace{g'' \frac{\partial f}{\partial x} \frac{\partial f}{\partial y}}_{?}$$

However, if this theorem holds for f, it holds for $g \circ f$, since if

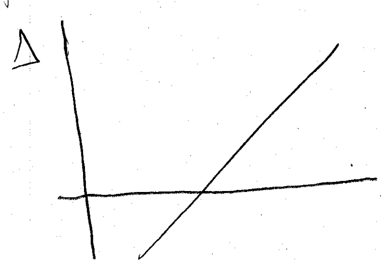
$x_f^*(t) = \arg\max_{x \in S(t)} f(x,t)$, then

$x_{f \circ g}^*(t) = x_f^*(t)$

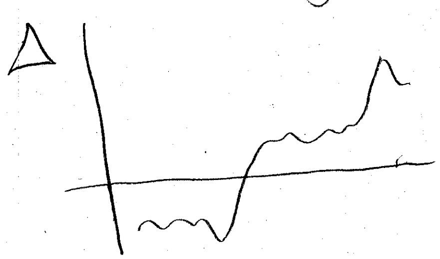
- Single-crossing: ordinal theory of complementarities
  - Milgrom-Shannon, Econometrica

E.g. 2 players:

- player 1: $a_1 \in \{0,1\}$
- player 2: $a_2 \in [0,1]$
- $\Pi_1(1,a_2) - \Pi_1(0,a_2) \equiv \Delta(a_2)$
  = value of investment for player 1
- complementarities: "I take a higher action when you take a higher action"
- supermodularity: $\Delta(\cdot)$ is increasing



- single-crossing: $\Delta(0)$ has exactly one sol'n.
  - much weaker condition.



II | Some facts on lattices and supermodular functions.

1] a finite lattice has a highest and a smallest element

- $(A, \geq)$, $\geq$ a partial order
- a_0. If $\exists$ a s.t. $a_0 \not\geq a$, let $a_1 = a \vee a_0$. Can proceed until all elements are exhausted.

2] Closed sublattices of complete lattices are complete.

- $(A, \geq)$ is complete if any subset has an inf and a sup in A.

3] If $(A, \geq)$ is complete, then $A \subset [x^L(A), x^H(A)]$, where $x^L(A)$ is the lowest element and $x^H(A)$ is the largest element.

- ie we can "summarize" a lattice by its extremal elements

4] What does $S \geq S'$ mean in terms of $x^h(S)$ and $x^h(S')$?

- If $S \geq S'$, then $x^h(S) \geq x^h(S')$ and $x^L(S) \geq x^L(S')$

Pf: $S \geq S' \Rightarrow x^H(S) \vee x^H(S') \in S \Rightarrow$

$$x^H(S) \geq x^H(S) \vee x^H(S') \geq x^H(S') \quad \square$$

Topkis' Thm: Let $f: X \times \mathbb{R} \to \mathbb{R}$ be supermodular in x (since $t \in \mathbb{R}$, this is equivalent to increasing differences in t and x). Let

$$x^*(t) = \underset{x \in S(t)}{\arg\max}\, f(x,t)$$

Then if $t \geq t'$ and $S(t) \geq S(t')$, then $x^*(t) \geq x^*(t')$

Pf: We want to show that if $x \in x^*(t)$ and $x' \in x^*(t')$, then $x \vee x' \in x^*(t)$. We want $x \vee x'$ maximizing $f(x,t)$ over $S(t)$.

1] $S(t) \geq S(t') \Rightarrow x \vee x' \in S(t)$

2] Since x maximizes f at t,

$$f(x,t) \geq f(x \vee x', t) \qquad (1)$$

Since x' maximizes f at t',

$$f(x',t') \geq f(x \wedge x', t') \qquad (2), \text{ since } x \wedge x' \in S(t')$$

$$\Rightarrow f(x \vee x', t) - f(x,t) \leq 0$$

and $0 \leq f(x',t') - f(x \wedge x', t')$

$\Rightarrow$ By increasing differences in t, we have

$$0 \geq f(x \vee x', t) - f(x,t) \geq f(x \vee x', t') - f(x,t')$$

$$\Rightarrow f(x \vee x', t') - f(x,t') \leq f(x',t') - f(x \wedge x', t') \quad (3)$$

If this inequality is strict, this contradicts f supermodular in x at t'.

Thus, $f(x,t) = f(x \vee x', t) \Rightarrow x \vee x'$ maximizes f at t over S(t).

How do you apply this to game theory?

Supermodularity $\Rightarrow$ monotone best reply

i.e. $(A_1, \geq_1), (A_2, \geq_2)$. If have monotone BR, then if $a_i \leq a_i'$, then $BR_{-i}(a_i) \leq_{-i} BR_{-i}(a_i')$

$\Rightarrow$ extreme Nash equilibria (exists a highest and lowest NE) and a nice way to compute them.

Milgrom-Roberts

$$\text{Define } x_i^*(s_{-i}) = \underset{s_i \in S_i}{\operatorname{argmax}} \overbrace{u_i(s_i, s_{-i})}^{\text{cf } f(x,t)}$$

If u_i is supermodular in s_i and increasing differences in s_{-i}, apply Topkis and have $x_i^*(s_{-i})$ is increasing in s_{-i}.

Note if u_i continuous in s_i, then $x_i^*(s_{-i})$ has a max and a min.

Since S_i is complete, $x_i^*(s_{-i})$ has a sup.
Since u_i is continuous, $x_i^*(s_{-i})$ is closed.

$$\Rightarrow \sup x_i^*(s_{-i}) \in x_i^*(s_{-i})$$

Define $BR_i^h(s_{-i}) = \max x_i^*(s_{-i})$

$BR_i^L(s_{-i}) = \min x_i^*(s_{-i})$

By Topkis, $BR_i^h(s_{-i})$ and $BR_i^L(s_{-i})$ are increasing.

Let $S_i = [\underline{x}_i, \bar{x}_i]$

$\Rightarrow \Omega \subset [\underline{x}, \bar{x}]$, where $\underline{x} = (\underline{x}_1, \dots, \underline{x}_n)$

$\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$

By monotone BR, we have:

$$BR(\Omega) \subset BR([\underline{x}, \bar{x}])$$

$$\underset{\text{monotone BR}}{\subseteq} [BR^L(\underline{x}), BR^h(\bar{x})] \subset \Omega$$

Then $\underline{x} \leq BR^L(\underline{x})$

$$\Rightarrow BR^L(\underline{x}) \leq \underbrace{BR^L(BR^L(\underline{x}))}_{= (BR^L)^2(\underline{x})} \leq \dots \leq (BR^L)^k(\underline{x}) \leq \dots$$

$BR^k(\Omega) \subseteq [(BR^L)^k(\underline{x}), (BR^h)^k(\bar{x})]$

$(BR^L)^k(\underline{x}) \to \underline{x}^\infty$, $(BR^h)^k(\bar{x}) \to \bar{x}^\infty$ with $\underline{x}^\infty, \bar{x}^\infty$ Nash.

and $BR^\infty(\Omega) \subseteq [\underline{x}^\infty, \bar{x}^\infty]$ if u_i continuous.