If $X = \mathbb{R}$, then $f$ is supermodular and submodular.

Pairwise supermodularity (increasing differences)

- Let $f: \mathbb{R}^N \to \mathbb{R}$. $f$ is <u>pairwise supermodular</u> iff $\forall n \neq m$ and $x_{-n,m}$, the restriction $f(\cdot, \cdot, x_{-nm}): \mathbb{R}^2 \to \mathbb{R}$ is supermodular.
- e.g. $f: \mathbb{R}^3 \to \mathbb{R}$
  - $(x,y,z)$, $(x',y',z')$
  - $$\left.\begin{aligned} & f(\max\{x,x'\}, \max\{y,y'\}, \max\{z,z'\}) \\ & + f(\min\{x,x'\}, \min\{y,y'\}, \min\{z,z'\}) \\ & \geq f(x,y,z) + f(x',y',z') \end{aligned}\right\} \text{supermodularity.}$$
- Take $z$ fixed: Then if:
  - $f(\max\{x,x'\}, \max\{y,y'\}, z) + f(\min\{x,x'\}, \min\{y,y'\}, z)$
    $\geq f(x,y,z) + f(x',y',z)$
  - similarly, if this holds if we fix $y$ and $x$, then $f$ is pairwise supermodular.

<u>Lemma</u>: If $f$ has increasing differences and $x_j \geq y_j$ for each $j$, then $f(x_i, x_{-i}) - f(y_i, x_{-i}) \geq f(x_i, y_{-i}) - f(y_i, y_{-i})$

Pf: $f(x_1, x_{-1}) - f(x_1, y_{-1})$ (fix $i=1$)

$= \sum_{j>1} f(x_1, x_2, \ldots, x_j, y_{j+1}, \ldots, y_n) - f(x_1, \ldots, x_{j-1}, y_j, \ldots, y_n)$

$$\geq \sum_{j>1} f(y_1, x_2, \dots, x_j, y_{j+1}, \dots, y_n) - f(y_1, x_2, \dots, x_{j-1}, y_j, \dots, y_n)$$

$$= f(y_1, x_{-1}) - f(y_1, y_{-1}). \square$$

e.g. $f(x_1, x_2, x_3) - f(x_1, y_2, y_3)$

$$= \underbrace{[f(x_1, x_2, x_3) - f(x_1, x_2, y_3)]}_{\geq} + \underbrace{[f(x_1, x_2, y_3) - f(x_1, y_2, y_3)]}_{\geq}$$

By increasing differences

$$\geq f(y_1, x_2, x_3) - f(y_1, x_2, y_3) \quad f(y_1, x_2, y_3) - f(y_1, y_2, y_3)$$

$$= f(y_1, x_2, x_3) - f(y_1, y_2, y_3)$$

<u>Theorem:</u> Let $f: \mathbb{R}^N \to \mathbb{R}$. Then $f$ is <u>supermodular</u> iff $f$ is pairwise supermodular

($\Rightarrow$) By definition

($\Leftarrow$)

$f(x \vee y) - f(y)$

$$= f(x_1 \vee y_1, \dots, x_n \vee y_n) - f(x_1 \vee y_1, \dots, x_{n-1} \vee y_{n-1}, y_n)$$
$$+ f(x_1 \vee y_1, \dots, x_{n-1} \vee y_{n-1}, y_n) - f(x_1 \vee y_1, \dots, x_{n-2} \vee y_{n-2}, y_{n-1}, y_n)$$

$$\vdots$$

$$+ f(x_1 \vee y_1, y_2, \dots, y_n) - f(y_1, \dots, y_n)$$

since, as a fn of $\mathbb{R}$, every function is supermodular and submodular (modular)

$$= \sum_{i>1} f(x_1 \vee y_1, \dots, x_{i-1} \vee y_{i-1}, x_i, y_{i+1}, \dots, y_n) - f(x_1 \vee y_1, \dots, x_{i-1} \vee y_{i-1}, x_i \wedge y_i, y_{i+1}, \dots, y_n)$$

$$\geq \sum_{i>1} f(x_1, \dots, x_{i-1}, x_i, x_{i+1} \wedge y_{i+1}, \dots, x_n \wedge y_n)$$
$$- f(x_1, \dots, x_i, x_i \wedge y_i, x_{i+1} \wedge y_{i+1}, \dots, x_n \wedge y_n)$$

$$= f(x) - f(x \wedge y) \; \square$$

Take $f: X_1, \ldots, X_n \to \mathbb{R}$. Under what conditions is $f$ supermodular?

- need $f(-, x_i \vee y_i, -) - f(-, y_i, -)$
  $\geq f(-, x_i, -) - f(-, x_i \wedge y_i, -)$

This replaces the second equality in the proof, so the theorem goes through.

- i.e. we need $\forall i$, $f|_{X_i}$ is supermodular and $f$ has increasing differences.

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"Pairwise" Sublattices

Thm: Let $S$ be a sublattice of $\mathbb{R}^N$. Define

$$S_{ij} = \{x \in \mathbb{R}^N \mid \exists z \in S \text{ s.t. } x_i = z_i, x_j = z_j\}$$

Then $S = \bigcap_{ij} S_{ij}$

Thus, in $\mathbb{R}^N$, we can express a sublattice as a collection of constraints on pairs of arguments. In particular $x_1 + x_2 + x_3 \leq 1$ cannot be decomposed in such a way, so this set is not a sublattice.

Thm: Let $f: \underbrace{X}_{\text{lattice}} \times \mathbb{R} \to \mathbb{R}$ be a supermodular function and define

$$x^*(t) \equiv \underset{x \in S(t)}{\arg\max}\, f(x,t)$$

∘ t, the parameter, affects both the function and the domain.

If $\underset{\uparrow}{t} \geq t'$, and $S(t) \geq S(t')$, then $x^*(t) \geq x^*(t')$

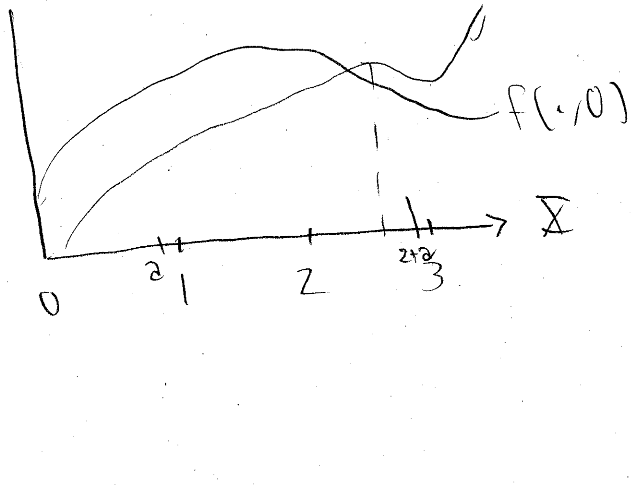
real number — strong set order

Recall (strong set order): $X \geq Y$ if $\forall x \in X, y \in Y$, $x \vee y \in X$ and $x \wedge y \in Y$.

Corollary: Let $f: X \times \mathbb{R} \to \mathbb{R}$ be a supermodular function and let $S(t)$ is a sublattice. Then for each $t$, $x^*(t)$ is a sublattice.

Pf: $t \geq t \Rightarrow S(t) \geq S(t)$ since $S(t)$ is a sublattice
$\Rightarrow x^*(t) \geq x^*(t)$ by thm. Thus, $x^*(t)$ is a sublattice.

E.g.



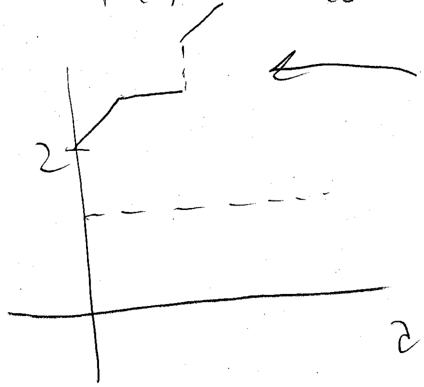

$t = 0, 1$

$S(0) = [0, 2]$

$S(1) = [a, a+2]$

Supermodularity: $\frac{\partial f(x,0)}{\partial x} < \frac{\partial f(x,1)}{\partial x}$

$f(\cdot, 0)$ has smaller derivative than $f(\cdot, 1)$




maxima from previous graph

Pf of thm:

- $[t \geq t', S(t) \geq S(t') \Rightarrow x^*(t) \geq x^*(t')]$ is what we want to show.
- Suppose $f$ is supermodular and that $x \in x^*(t)$, $x' \in x^*(t')$, $t > t'$. We want to show that $x \vee x' \in x^*(t)$ and $x \wedge x' \in x^*(t)$
- Since $x \in x^*(t)$, $x \in S(t)$
  Since $x' \in x^*(t')$, $x' \in S(t')$

We also know that $S(t) \geq S(t')$

Thus, $x \vee x' \in S(t)$ and $x \wedge x' \in S(t')$

Thus, $f(x,t) \geq f(x \vee x', t)$ and $f(x',t') \geq f(x \wedge x', t')$, since $x \vee x'$ is feasible under $t$, but $x$ is optimal. Similarly for $t'$.

Want to show $f(x \vee x', t) = f(x, t)$, so that $x \vee x' \in x^*(t)$. Similarly for $f(x', t') = f(x \wedge x', t')$

Suppose one of these inequalities is strict.
Then $f(x,t)+f(x',t') > f(x \vee x',t)+f(x \wedge x',t')$
$\Rightarrow f$ is not supermodular $\rightarrow\!\!\leftarrow$