Supermodularity

In economics, we are interested in comparative statics.

- Typically, we use calculus
- Here, we will think about this more generally: lattice theory.

Complementarity:

- in terms of constraints
  - two activities are complementary if doing one does not preclude doing the other.
  - "increasing one component makes the other one easier to increase."
- in terms of payoffs
  - two activities are complementary if doing one makes the other weakly more profitable.
  - this is described by supermodularity of payoffs

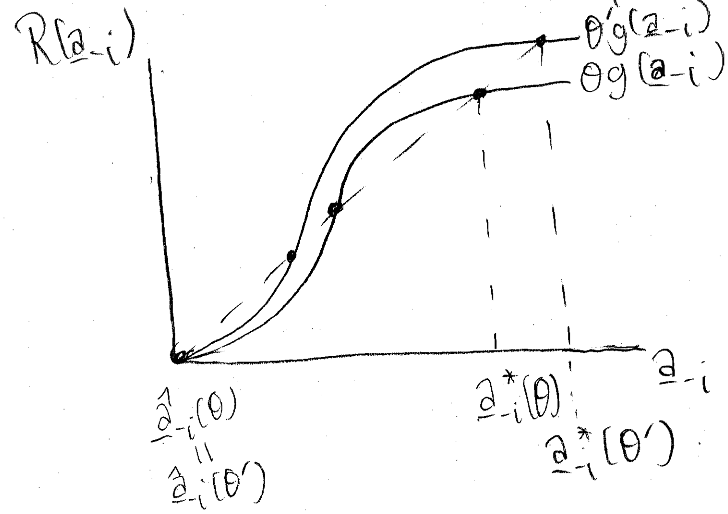
Peter Diamond search model.

- continuum of players
  - prob. of being matched increases in effort a_i
- cost of search is $a_i^2/2$

- Prob i finds match: $a_i g(\underline{a}_{-i})$, where $\underline{a}_{-i}$ is average of everyone else's effort.
- payoff if matched: θ
- $U_i(a) = \theta a_i g(\underline{a}_{-i}) - \frac{a_i^2}{2}$

  FOC wrt (a_i): $\theta g(\underline{a}_{-i}) = a_i^* = R(\underline{a}_{-i})$

  - strategic complementarity: if $\underline{a}_{-i}$ increases, want to increase a_i^*: $R(\underline{a}_{-i}) = \theta g(\underline{a}_{-i})$



- the maximum equilibrium is increasing in θ.
- the minimum equilibrium is also increasing in θ.

(*) Supermodularity and network externalities.

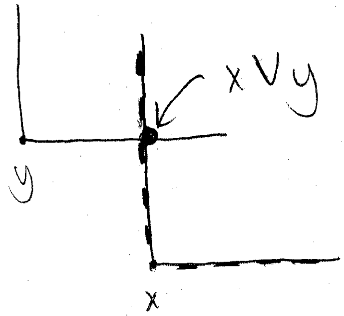
Let $(X, \geq)$ be partially ordered. Let $x, y \in X$.

Define the join of x and y by:

- $x \vee y = \inf\{z \in X : z \geq x, z \geq y\}$

Define the meet of x and y by:

- $x \wedge y = \sup\{z \in X : z \leq x, z \leq y\}$



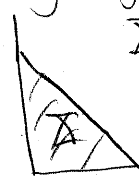
$y=(0,2)$, $x=(1,0)$ } $x \vee y = (1,2)$, $x \wedge y = (0,0)$

$(X, \geq)$ is a <u>lattice</u> if it is closed under meet and join. i.e. $\forall x, y \in X$, $x \vee y \in X$ and $x \wedge y \in X$.

e.g. Let $X = \mathbb{R}^n$. Take $X \ni x = (x_1, \dots, x_n)$, $X \ni y = (y_1, \dots, y_n)$

Then $x \wedge y = (\min\{x_1, y_1\}, \dots, \min\{x_n, y_n\})$

$x \vee y = (\max\{x_1, y_1\}, \dots, \max\{x_n, y_n\})$



X is not a lattice under the standard order.

<u>Defn</u>: We say $(X, \geq)$ is a <u>complete lattice</u> if $\forall$ nonempty subset S, $\inf(S) \in X$, $\sup(S) \in X$. (and $(X, \geq)$ is a lattice)

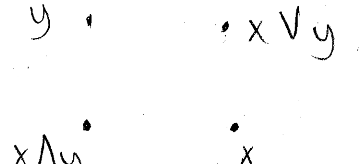
<u>Defn</u>: a function $f: X \to \mathbb{R}$ is <u>supermodular</u> if
$\forall x, y \in X$, $f(x) + f(y) \leq f(x \wedge y) + f(x \vee y)$

∘ f is <u>submodular</u> if $-f$ is supermodular

Equivalently,

∘ f is supermodular if $f(x \vee y) - f(x) \geq f(y) - f(x \wedge y)$

or $f(x \vee y) - f(y) \geq f(x) - f(x \wedge y)$



∘ increasing marginal returns

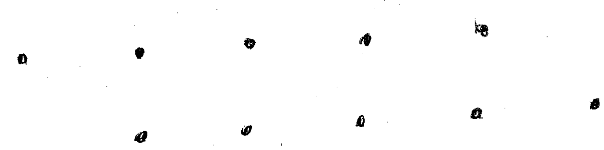
$\Leftrightarrow$ f is supermodular if $[f(x \vee y) - f(x)] - [f(y) - f(x \wedge y)] \geq 0$

For smooth functions, need $\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$ for $i \neq j$

<u>Defn</u> Given two subsets $S, T \subseteq X$, we say S is <u>as high as</u> T $(S \geq T)$ if $\forall x \in S, y \in T$, $x \vee y \in S$ and $x \wedge y \in T$. (Strong set order)

e.g. S_1

S_2



$S_2 \geq S_1$

e.g. $S_1 = \{0,1,2\}$
$S_2 = \{1,2,3\}$

$$\left.\begin{array}{l}
0 \vee 1 = 1 \in S_2,\ 0 \vee 2 = 2 \in S_2,\ 0 \vee 3 = 3 \in S_2 \\
1 \vee 1 = 1 \in S_2,\ 1 \vee 2 = 2 \in S_2,\ 1 \vee 3 = 3 \in S_2 \\
2 \vee 1 = 2 \in S_2,\ 2 \vee 2 = 2 \in S_2,\ 2 \vee 3 = 3 \in S_2 \\
0 \wedge 1 = 0 \in S_1,\ 0 \wedge 2 = 0 \in S_1,\ 0 \wedge 3 = 0 \in S_1 \\
1 \wedge 1 = 1 \in S_1,\ 1 \wedge 2 = 1 \in S_1,\ 1 \wedge 3 = 1 \in S_1 \\
2 \wedge 1 = 1 \in S_1,\ 2 \wedge 2 = 2 \in S_1,\ 2 \wedge 3 = 2 \in S_1
\end{array}\right\} S_2 \geq S_1$$

e.g. $S_1 = \{0,1\}$
$S_2 = \{\frac{1}{2}, \frac{3}{2}\}$

Neither $S_1 \geq S_2$ nor $S_2 \geq S_1$

<u>Defn</u>: a function x^* is <u>isotone</u> if $t \geq t' \Rightarrow x^*(t) \geq x^*(t')$

<u>Defn</u>: a set S is a <u>sublattice</u> if $S \geq S$. (ie a subset of a lattice which is closed under meet and join).

Let $X \subseteq 2^A$ for some set A.
$(X, \geq)$, where $x \geq y$ if $x \supseteq y$
$x \vee y = x \cup y$, $x \wedge y = x \cap y$.