Fudenberg - Levine

a repeated game $G(\delta)$
- 2 players, finite actions
- perfect monitoring

Player 1 is LR with
- normal type θ_0 with payoff as in G: $p(\theta_0) > 0$
- commitment types, θ, with dominant strategies in repeated game: $p(\theta) > 0$

Player 2 is played by series of SR players
- $BR_2(a_1)$, $BR_2(\underbrace{\alpha_1}_{\text{if believes mixed}})$

Crucial Lemma:
- Consider any θ that always plays a_1^* as its dominant strategy.
- Fix any strategy profile $\sigma = (\sigma_1, \sigma_2)$
- h^t = a history with P1 always playing a_1^*
- $\pi(h^t) = Pr[a^{t+1} = a_1^* \mid h^t, \sigma_1]$

$$= 1 \cdot \mu(\theta \mid h_t) + (1 - \mu(\theta \mid h_t))\, \sigma_1(a_1^* \mid h_t)$$

(if only two types)

Then, for any $q < 1$,
- $K(q) = \# \{t \mid \pi(h^t) \leq q\} \leq \ln(p(\theta)) / \ln q$

Pf: $\underset{\substack{\text{type that} \\ \text{plays } a_1^* \\ \text{always}}}{\mu(\theta|h^t, a^{t+1})} = \mu(\theta|h^t, (a_1^*, a_2^{t+1}(h^t)))$

$\Pr[a^1 = a_1^* | \theta]$

$$= \frac{1 \cdot \sigma^2(a_2^{t+1}(h^t))\, \mu(\theta|h^t)}{\sigma^2(a_2^{t+1}(h^t))\, \pi(h^t)}$$

$$= \frac{\mu(\theta|h^t)}{\pi(h^t)}$$

$$1 \geq \mu(\theta|h^\infty) \geq \frac{p(\theta)}{q^{K(q)}}$$

$$p(\theta) \cdot \frac{1}{\pi(h^0)} \cdot \frac{1}{\pi(h^1)} \cdot \ldots = \mu(\theta|h^\infty)$$

Let $K(q) = \#\{t \mid \pi(h^t) \leq q\}$

$$\Rightarrow \pi(h^0)\, \pi(h^1) \cdots \leq \underbrace{q \cdot q \cdots q}_{K(q)} \cdot 1 \cdot 1 \cdots = q^{K(q)}$$

$$\Rightarrow \mu(\theta|h^\infty) \geq \frac{p(\theta)}{q^{K(q)}}$$

$$\Rightarrow \ln(1) = 0 \geq \ln p(\theta) - K(q) \ln q$$

$$\Rightarrow K(q) \ln q \geq \ln p \Rightarrow K(q) \leq \frac{\ln p(\theta)}{\ln q}$$

since $\ln q < 0$

Theorem:

- $C = \{a_1 \mid \text{playing } a_1 \text{ is dominant for some type } \theta\}$

For any $\lambda < 1$, $\exists\, \underline{\delta} < 1$ s.t. $\forall \delta > \underline{\delta}$, $\forall$ NE σ^*,

$$u_1(\sigma^*) \geq (1-\lambda) \min_a u_1(a) + \lambda \underbrace{\max_{a_1 \in C} \min_{a_2 \in BR_2(a)} u_1(a_1, a_2)}_{\text{constrained maxmin (Stackelberg)}}$$

With incomplete information, many of the NE go away

Proof:

- max is obtained in a_1^* played by θ
- h^t = a history where P1 always plays a_1^*
- $\exists\, q < 1$ s.t. $\pi(h^t) > q \Rightarrow \sigma_2^*(h^t) \in BR_2(a_1^*)$ by upper-hemicontinuity

Then, payoff from always playing a_1^* against σ_2^* is greater than or equal to:

$$(1-\delta^{K(q)}) \min_a u_1(a) + \delta^{K(q)} \max_{a_1 \in C} \min_{a_2 \in BR_2(a_1)} u_1(a_1, a_2)$$

Next, $\delta^{K(q)} \geq \delta^{\frac{\ln p(\theta)}{\ln q}} > \lambda$. take $\underline{\delta}$ such that

$$\underline{\delta} = \lambda^{\ln(q)/\ln p(\theta)}$$

Characterization Theorem

Stackelberg Payoff:

- $U^s \equiv \max_{\alpha_1} \max_{\alpha_2 \in BR_2(\alpha_1)} u_1(\alpha_1, \alpha_2)$

"another Stackelberg payoff":

- $U^* \equiv \max_{\alpha_1} \min_{\alpha_2 \in BR_2(\alpha_1)} u_1(\alpha_1, \alpha_2)$

assume:

- mixed actions are observed
- playing Stackelberg always is dominant for some type.

Then $\forall \lambda < 1, \exists \underline{\delta} < 1$ s.t. $\forall \delta > \underline{\delta}$ and every NE σ^*,

$$(1-\lambda)\min_a u_1(a) + \lambda U^* \le U_1(\sigma^*) \le (1-\lambda)\max_a u_1(a) + \lambda U^s$$

When $\delta \to 1$, can take $\lambda \to 1$ and

$$U^* \le U_1(\sigma^*) \le U^s \underset{\substack{\uparrow \\ \text{for generic games}}}{=} U^*$$

Thus, $U_1(\sigma^*) = U^*$. Unique player 1 payoffs over the entire set of NE.

Folk theorem collapses into a single point.