Repeated Games w/ imperfect public monitoring

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Public perfect equilibrium and self-generation.

Defn: Stage game $A = (N, A, u)$ simultaneous

- A finite or A convex and compact
- $a \in A$ leads to publicly signal $y \in Y$, where
  $Pr[y|a] = p(y|a)$
- $u(a) = \sum_{y \in Y} u^*(a,y) \, p(y|a)$

Repeated game: at each $t = 0, 1, 2, \ldots$

- players play $a^t \in A$
- public signal $y^t \in Y$ realized
- only $(y^0, \ldots, y^{t-1})$ is observable at t.
  - If $(y^0, \ldots, y^{t-1}) = (a^0, \ldots, a^{t-1})$, then this is a game of perfect information
- payoffs: $U_i = (1-\delta) \sum_{t=0}^{\infty} \delta^t u_i(a^t)$
- A convex, compact and u^*, p continuous and u_i qc in $a_i \Rightarrow$ unique BR.

- Public history $h^t = (y^0, \ldots, y^{t-1})$
- Private histories include own actions
- $s = (s_1, \ldots, s_n)$ strategy profile, $\sigma = (\sigma_1, \ldots, \sigma_n)$ mixed
- Mixed action profile $\alpha \in \prod_{i=1}^{n} \Delta(A_i)$

- V - set of feasible payoffs; compact
- Strategy σ_i (or s_i) is a <u>public</u> strategy if σ_i^t depends only on public history h^t.

<u>Lemma</u>: If everyone plays public strategies, it is a best reply to play a public strategy. (since $p(y|a)$ is iid over time)

<u>Lemma</u>: Every pure strategy is equivalent to a pure public strategy

- a PPE is a public strategy profile that specifies a NE after every h^t
- V^{PPE} = payoffs from all PPE.

<u>Public strategies as automata</u>

Public automaton of $RG(\delta)$ is a
- set Q of states
- q^0 in Q
- Output function $f: Q \to \prod_{i=1}^{n} \Delta(A_i)$

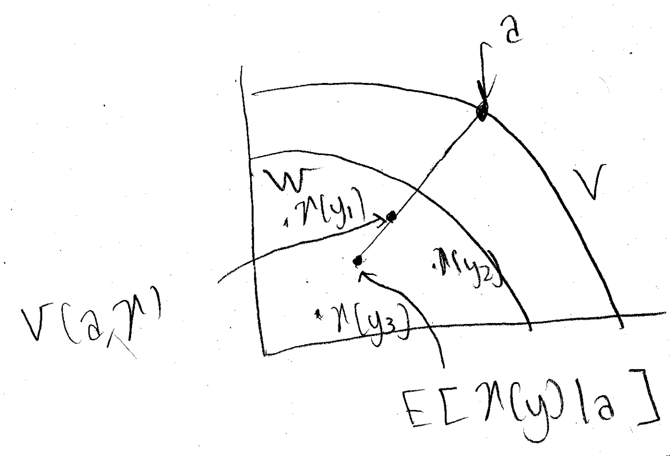
- transition function $\tau: Q \times Y \to Q$

<u>Thm (SDP)</u>: For any automaton (Q, q^0, f, τ), let $U(q)$ be a vector of continuation values at q. Strategy profile induced by (Q, q^0, f, τ) is a PPE iff $\forall$ accessible q, $f(q)$ is a NE of (N, A, g^q), where

- $g^q(a) = (1-\delta)u(a) + \delta \sum_{y \in Y} U(\tau(q,y))\rho(y|a)$

<u>Self generation</u>

- For any $W \subseteq \underbrace{V}_{\text{feasible payoff profiles}}$, mixed action profile α is enforceable on W if $\exists\, \gamma: Y \to W$ s.t. α is a NE of (N, A, V), where
  - $V(a, \gamma) = (1-\delta)u(a) + \delta E[\gamma(Y)|a]$



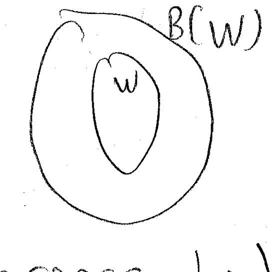
$1/3$	$1/3$	$1/3$
y_1	y_2	y_3
$\downarrow$	$\downarrow$	$\downarrow$
$\gamma(y_1)$	$\gamma(y_2)$	$\gamma(y_3)$

- machine gives payoffs only in W

Define $B(W) = \{V(a,\gamma) : \alpha \text{ enforced by } \gamma \text{ on } W\}$

- feasibility of mechanism design problem

- W is <u>self generating</u> if $W \subseteq B(W)$
  - If tomorrow, I have access to W, then today I have access to $B(W)$, and hence I still have access to W.



<u>Thm</u>: If $W \subseteq V$ is self-generating, then $W \subseteq B(W) \subseteq$ UPPE.

- ie anything in $B(W)$ can be supported by UPPE.

<u>Pf</u>: $W \subseteq B(W) \Rightarrow B(W) \subseteq$ UPPE (ie $\forall v^* \in B(W), \exists \sigma \in PPE$ s.t. $v^* = U(\sigma)$)

$\gamma \in W^Y$

- Define $f: B(W) \to \prod_{i=1}^{n} \Delta(A_i)$ and $\gamma: Y \to W$ such that $v = V(f(v), \gamma(v))$ and $\gamma(v)$ enforces $f(v)$ on W.
- For each v^* in $B(W)$, consider public automaton $(B(W), \underbrace{v^*}_{\text{initial state}}, f, \tau)$ with $\tau(v,y) = \gamma(v)(y)$
- Continuation vector U is the unique fixed point of contraction mapping $F: X \mapsto F(X)$, $X: B(W) \to V$

$$F(X)(v) = (1-\delta)u(f(v)) + \delta E[X(\tau(v,y))].$$

  - $U(v) = (1-\delta)u(f(v)) + \delta E[U(\tau(v,y))]$

- $U(v) = v$ is a fixed point of F by inspection.

$\Rightarrow$ This automaton ensures that at v, everyone gets v.

SDP: $(B(W), v^*, f, \tau)$ is a PPE, $\forall v \in B(W)$,
$f(v)$ is a NE of (N, A, g^v) with

- $g^v(a) = (1-\delta)u(a) + \delta E[\tau(v,y)|a]$
$= (1-\delta)u(a) + \delta E[\gamma(v)(y)|a] = V(a, \gamma(v))$ $\square$

Thm: 1] UPPE is the largest self-generating W.
2] $B(\text{UPPE}) = \text{UPPE}$