Simple strategies

a simple strat. profile $\sigma(a(0), a(1), \ldots, a(n))$ associated with outcome paths $a(0), \ldots, a(n)$:

- $Q = \underbrace{\{0, 1, \ldots, n\}}_{\text{paths}} \times \underbrace{\{0, 1, 2, \ldots\}}_{\text{time}}$
- $q^0 = (0, 0)$
- $f(j,t) = a^t(j)$
- $\tau((j,t), a) = \begin{cases} (i, 0) & \text{if } a_i \neq a_i^t(j) \text{ and } a_{-i} = a_{-i}^t(j) \\ (j, t+1) & \text{otherwise} \end{cases}$

- Each path is associated with a player i

$a(1)$ $\quad a^0(1)$ $\quad a^1(1)$ $\quad a^2(1)$ $\ldots$ (1's punishment path)

if 1 deviates

$a(0)$ $\quad a^0(0)$ $\quad a^1(0)$ $\quad a^2(0)$ $\ldots$ (equilibrium path)

if 2 deviates

$a(2)$ $\quad a^0(2)$ $\quad a^1(2)$ $\quad a^2(2)$ $\ldots$ (2's punishment path)

if 2 deviates

- If both deviate, stay where you are

- Let $U_i^t(a) = (1-\delta)\sum_{\tau \geq t} \delta^{\tau - t} u_i(a^t)$
- $\sigma(a(0), a(1), \dots, a(n))$ is SPE iff $\forall i, j, t$

$$U_i^t(a(j)) \geq \max_{a_i \in A_i} (1-\delta) u_i(a_i, a_{-i}^t(j)) + \delta U_i^0(a(i))$$

Optimal Penal Codes

- Consider simple strategies that yield worst payoffs.
- The set of pure SPE is compact if the game is continuous at infinity.
- assume it is nonempty.

Write: $v_i^* = \underbrace{\min}_{\text{not infimum}} \{U_i(\sigma) \mid \sigma \text{ is pure SPE}\}$

For outcome paths $a(1), \dots, a(n)$, an optimal penal code is $(\sigma(1), \dots, \sigma(n))$ with

- $\forall i, \sigma(i) = \sigma(a(i), a(1), \dots, a(n))$ is a pure SPE
- $\forall i\ U_i^0(a(i)) = v_i^*$ • ie everyone gets worst punishment if deviate

Thm: Let $(a(1), \dots, a(n))$ be outcome paths of pure SP equilibria $(s(1), \dots, s(n))$ with $U_i^0(a(i)) = v_i^* \; \forall i$.
Then:

- $(\sigma(1), \dots, \sigma(n))$ for $a(1), \dots, a(n)$ is an optimal penal code.
- $a(0)$ is an outcome path of a pure SPE, say $s(0)$ iff $\sigma(a(0), a(1), \dots, a(n))$ is an SPE.

Steps: $\boxed{1}$ Find worst SPE for each player i
$\boxed{2}$ Use those to punish players

To prove, note that $s(j)$ is SPE $\forall j = 0, 1, \dots, n$. Want to show that $\sigma(a(j), a(1), \dots, a(n))$ is SPE $\forall j$.

Since $s(j)$ is SPE,

$$U_i^t(a(j)) \geq \max_{a_i} (1-\delta) u_i(a_i, a_{-i}^t(j)) + \delta \underbrace{U_i(s \mid h^t, a_i, a_{-i}^t(j))}_{\geq v_i^* = U_i^0(a(i))}$$

$$\geq \max_{a_i} (1-\delta) u_i(a_i, a_{-i}^t(j)) + \delta U_i^0(a(i)) \; \text{I}$$

Folk Theorem:

Feasibility and individual rationality

Let $\underline{v}_i = \min_{\alpha_{-i} \in \prod_{j \neq i} \Delta(A_j)} \max_{a_i \in A_i} u_i(a_i, \alpha_{-i})$

	p H	1-p T
H	1,-1	-1,1
T	-1,1	1,-1

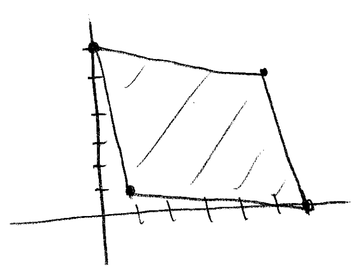
$H \rightarrow 2p-1$

$T \rightarrow 1-2p$

$$\min_p \max \{2p-1, 1-2p\} = 0 \text{ if } p = \tfrac{1}{2}.$$

- Let $m^i = (m_1^i, \ldots, m_n^i)$ - minmax profile for i.
  - a payoff profile v is <u>feasible</u> if it is a convex combination of payoff profiles $u(a)$ with $a \in A$
  - a payoff profile v is [strictly] <u>individually rational</u> if $[v_i > \underline{v}_i]$ $v_i \geq \underline{v}_i$ $\forall i$.
  - a is ind. rational if $u(a)$ is ind. rational

5,5	0,6
6,0	1,1



Prop: Every NE payoff profile (in repeated and stagegame) is individually rational

Thm: Consider a^* which is a strictly IR action profile. assume that m^i is pure for each i. assume $\exists$ strictly IR profiles $\underbrace{a(1), \ldots, a(n)}_{\text{punishments}}$ s.t. $\forall i$

$u_i(a^*) > u_i(a(i))$ and $u_i(a(j)) > u_i(a(i))$

for $j \neq i$.

Then $\exists$ $\underline{\delta} < 1$ s.t. $\forall \delta \in (\underline{\delta}, 1)$ $\exists$ a SPE s of $RG(\delta)$ in which a^* is played on the equilibrium path.

Want to show $\forall i, j, t$ $U_i^t(a(j)) \geq \max_{a_i} (1-\delta) u_i(a_i, a_{-i}^t(j)) + \delta U_i^0(a(i))$

Pf: [1] Take $\underbrace{K}_{\text{punishment length}}$ s.t. $\forall i$ $\max_a u_i(a) + K\underline{v}_i < \min_a u_i(a) + K u_i(a(i))$

[2] Consider simple strat. profile $\sigma(\mathbf{a}(0), \mathbf{a}(1), \ldots, \mathbf{a}(n))$ s.t.

- $\mathbf{a}(0) = (a^*, a^*, \ldots)$
- $\mathbf{a}(i) = (\underbrace{m^i, m^i, \ldots, m^i}_{K \text{ times}}, a(i), a(i), \ldots)$

[3] $U_i^t(\mathbf{a}(0)) = u_i(a^*)$

[4] $U_i^t(\mathbf{a}(j)) = \begin{cases} (1-\delta^{K-t+1}) u_i(m^j) + \delta^{K-t+1} u_i(a(j)) & \text{if } t \leq K \\ u_i(a(j)) & \text{otherwise} \end{cases}$

5] Can pick $\underline{\delta}$ s.t. $\forall\ \underline{\delta} < \delta < 1$ $\forall\ i, j, t$,

$$U_i^t(a(j)) \geq \max_{a_i \in A_i} (1-\delta) u_i(a_i, a_{-i}^t(j)) + \delta U_i^0(a(i))$$

e.g: $u_i(a^*) \geq \underbrace{(1-\delta) \max_a u_i(a) + \delta U_i^0(a(i))}$

$\rightarrow U_i^0(a(i))$ as $\delta \rightarrow 1$,

so that $u_i(a^*) \geq (1-\delta^*) \max_a u_i(a) + \delta^* U_i^0(a(i))$

for some $\delta^* < 1$.

$i \neq j \neq 0$, $t \leq K$

- $U_i^t(a(j)) = (1-\delta^{K-t+1}) u_i(m^j) + \delta^{K-t+1} u_i(a(j))$

$\rightarrow u_i(a(j))$ as $\delta \rightarrow 1$

If deviate: $(1-\delta) \max_{a_i} u_i(a) + \delta U_i^0(a(i))$

$\rightarrow u_i(a(i))$ as $\delta \rightarrow 1$

Since $u_i(a(j)) > u_i(a(i))$, $\exists\ \delta_{ijt}$ s.t. $\delta > 1$

$\Rightarrow$ Not want to deviate.

$i=j$: Want to show:

$$U_i^0(a(i)) \geq \overbrace{\max_{a_i} \underbrace{(1-\delta) u_i(a_i, m_{-i}^i)}_{\text{at } t=0}}^{\underline{v}_i^*} + \delta U_i^0(a(i))$$

$\Leftrightarrow U_i^0(a(i)) \geq \underline{v}_i^*$ which holds by assumption