Repeated Games w/ Perfect Monitoring

Stage game $G=(N, A, u)$
- finite, simultaneous move game

Let $RG(\delta)$ be a multistage game
- Stages $t=0,1,2,\ldots$ G is played at each t
- $a^t \in A$ played at t
- $U_i = (1-\delta)\sum_{t=0}^{\infty} \delta^t u_i(a^t)$

Notation:

- History: $h^t=(a^0, a^1, \ldots, a^{t-1})$
- Clearly, $h^{t+1}=(h^t, a^t)=(a^0, a^1, \ldots, a^{t-1}, a^t)$
- Initial history h^0
- Outcome path $a=(a^0, a^1, \ldots)$
- $s=(s_1, \ldots, s_n)$ pure strategy profile
- $\sigma=(\sigma_1, \ldots, \sigma_n)$ mixed strategy profile
- $\sigma^t(h^t)$ mixed action profile at h^t according to σ
- $\sigma|h^t$ continuation of σ after h^t
- $U(\sigma|h^t)$ vector of continuation values from σ at h^t.

SDP for static stage game

	C	D
C	5, 5	0, 6
D	6, 0	1, 1

- a strategy profile σ is SPE iff
- $\forall h^t$ $\forall i$,
  - at t and thereafter, $j \neq i$ plays according to σ
  - at $t+1$ and thereafter, $\forall i$, i plays according to σ

Then i does not want to deviate at h^t.

Grim: C iff $\nexists$ D before

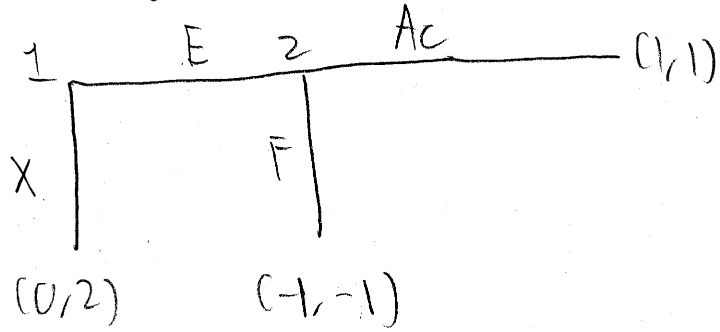
(Grim, Grim) SPE?

- $(C, C, \ldots, C) \Rightarrow$ if play C, get $(1-\delta)5 + 5\delta = 5 \equiv V_c$

  if play D, get $(1-\delta)6 + \delta = 6 - 5\delta \equiv V_{D|C}$

  - need $V_c \geq V_{D|C} \iff 5 \geq 6 - 5\delta \iff \delta \geq \frac{1}{5}$
- Similarly for if deviate ever occurs

SDP for dynamic stage games



- Let a^t be the "path of play of G" at t

  $a^t \in \{X, EAc, EF\}$

- For $h^t = (a^0, a^1, \ldots, a^{t-1})$, define $G(h^t, \sigma) = (N, A, v)$, where $\forall a \in A$,

$$v(a) = (1-\delta)u(a) + \delta U(\sigma \mid (h^t, a))$$

- σ is SPE iff $\forall h^t$, $\sigma^t(h^t)$ is a SPE of $G(h^t, \sigma)$

Consider:

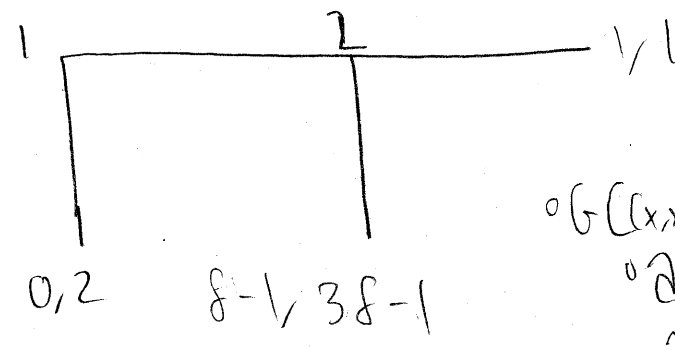
- $\{E \text{ iff } \nexists Ac,\ Ac \text{ iff } \nexists Ac\} = \sigma$

Suppose $\exists$ A_c in the past (e.g. $h^t = (x, x, x, EF, x, \ldots, x)$)

- payoffs: $U_1 = (1-\delta)0 + \delta 0 = 0$ if not enter
  $U_2 = (1-\delta)2 + \delta 2 = 2$
- if EF: $U_1 = (1-\delta)(-1) + \delta 0 = \delta - 1$
  $U_2 = (1-\delta)(-1) + 2\delta = 3\delta - 1$
- if EA_c: $U_1 = (1-\delta)1 + \delta 1 = 1$
  $U_2 = (1-\delta)1 + \delta 1 = 1$

augmented game

Need
- $3\delta - 1 \geq 1 \iff \delta \geq \frac{2}{3}$
- $0 \geq \delta - 1 \iff \delta \leq 1$



- $G((x, x, x, EF, x, \ldots, x), \sigma)$
- augmented game after h^t, σ.

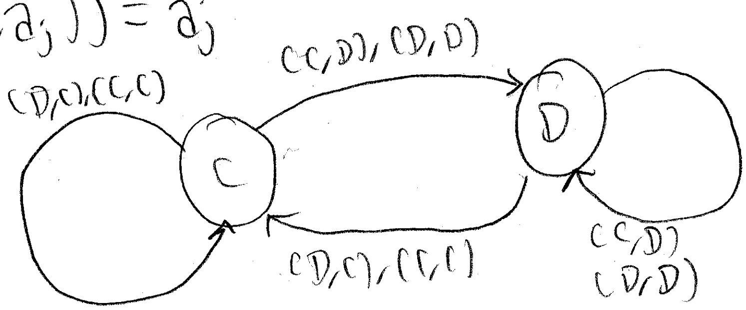
Strategies as automata

an automata for i in $RG(\delta)$ consists of:

- a set Q_i of states
- an initial state $q_i^0 \in Q_i$
- an output function $f_i : Q_i \to \Delta(A_i)$
  - at each state q_i, $f_i(q_i)$ tells you which mixed action to play
- a transition function $\tau_i : Q_i \times A \to Q_i$

e.g. Tit-for-tat

- C at beginning. afterwards play what other person played
- $Q_i = \{C, D\} = \{\text{what other person did}\}$
- $q_i^0 = C$
- $\tau_i(C) = C$, $\tau_i(D) = D$
  - $\tau_i(q_i, (a_i, a_j)) = a_j$
- $f_i(q_i) = q_i$



- Want to write s_i as an automaton:
  - $Q_i = \{h^t : t < +\infty\}$
  - $q_i^0 = h^0$
  - $\tau_i(q_i, a) = (q_i, a)$
    - i.e. $q_i = (a^0, \ldots, a^{t-1})$

$$\tau_i(q_i, a) = (a^0, \ldots, a^{t-1}, a)$$

  - $f_i(h^t) = s_i(h^t)$

The strategy σ_i induced by $(Q_i, q_i^0, f_i, \tau_i)$:

- h^0: $\sigma_i(h^0) = f_i(q_i^0)$
- $h^1 = (a^0)$:
  - $q_i^1(h^1) = \tau_i(q_i^0, a^0)$
  - $\sigma_i(h^1) = f_i(q_i^1(h^1))$
- $h^2 = (a^0, a^1)$:
  - $q_i^2(h^2) = \tau_i(q_i(a^0), a^1)$
  - $\sigma_i(h^2) = f_i(q_i^2(h^2))$

$\vdots$

- $h^t = (a^0, \ldots, a^{t-1})$:
  - $q_i^t(h^t) = \tau_i(q_i(a^0, \ldots, a^{t-2}), a^{t-1})$
  - $\sigma_i(h^t) = f_i(q_i^t(h^t))$

a strategy profile σ can be represented

by a grand automaton (Q, q^0, f, τ) where
$f : Q \to \prod_{i=1}^{n} \Delta(A_i)$

- e.g. (Tit-for-tat, Tit-for-tat)
  - $Q = A = \{(C,C), (C,D), (D,C), (D,D)\}$
  - $q^0 = (C,C)$
  - $\tau(q, (a_1, a_2)) = (a_2, a_1)$
  - $f(q) = q$

q is accessible from q^0 if $\exists\, h^t$ with
$q^t(h^t) = q$

SDP for automata

- Given any (Q, q^0, f, τ), let $V(q^0)$ be the payoff profile from the strategy profile induced by (Q, q^0, f, τ)
- Strat. prof. induced by (Q, q^0, f, τ) is SPE iff $\forall$ states q accessible from q^0, $f(q)$ is a NE of (N, A, v), where
  $v(a) = (1-\delta)u(a) + \delta V(\tau(q, a))$

- $V((C,C)) = (5,5)$
- $V((C,D)) = \left(\frac{6\delta}{1+\delta}, \frac{6}{1+\delta}\right)$
- $V((D,C)) = \left(\frac{6}{1+\delta}, \frac{6\delta}{1+\delta}\right)$
- $V((D,D)) = (1,1)$

For (C,C):

	C	D
C	$(1-\delta)5+\delta5=5$ $(1-\delta)5+\delta5=5$	$(1-\delta)0+\delta\left(\frac{6}{1+\delta}\right)$ $(1-\delta)6+\delta\left(\frac{6\delta}{1+\delta}\right)$
D	$(1-\delta)6+\delta\left(\frac{6\delta}{1+\delta}\right)$ $(1-\delta)0+\delta\left(\frac{6}{1+\delta}\right)$	$(1-\delta)1+\delta1=1$ $(1-\delta)1+\delta1=1$

- $V(C,C)=5 \geq (1-\delta)6+\frac{\delta^2}{1+\delta}6=\frac{6}{1+\delta}$

For (D,C)

	C	D
C	$(1-\delta)5+5\delta=5$	
D	$\frac{6}{1+\delta}$ $\frac{6\delta}{1+\delta}$	

- Need $\frac{6}{1+\delta} \geq 5 \iff 5+5\delta \leq 6$

$$5\delta \leq 1$$
$$\delta \leq \frac{1}{5}$$

a <u>simple strategy profile</u> $\sigma(a(0), a(1), \ldots, a(n))$ associated with outcome paths $a(0), \ldots, a(n)$:

- $Q = \{0, 1, 2, \ldots, n\} \times \underbrace{\{0, 1, 2, \ldots\}}_{\text{corresponds to time}}$
- $q^0 = (0, 0)$
- $f(j, t) = a^t(j)$
- $\tau((j,t), a) = \begin{cases} (i, 0) & \text{if } a_i \neq a_i^t(j) \text{ and } a_{-i} = a_{-i}^t(j) \\ (j, t+1) & \text{otherwise} \end{cases}$

$a(1)$ $(D,C), (D,C) \ldots$

if 1 deviates

$a(0)$ $(C,C), (C,C) \ldots$

if 2 deviates

$a(2)$ $(C,D), (C,D) \ldots$