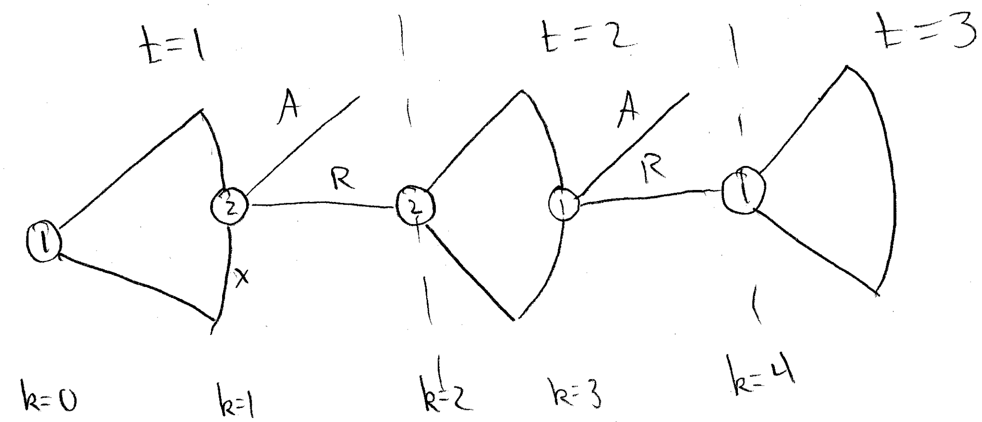
Recall from last time:




Notation and definitions

- $h = (a^0, a^1, \ldots, a^{k-1}, a^k, \ldots)$
- $h^k = (a^0, a^1, \ldots, a^{k-1})$
- $G(k,h)$ - subgame starting at $k$ with $h = (a^0, \ldots, a^{k-1})$ (generic history, can think of as $h^k$)
- $b^{k,h}$ = restriction of behavioral strategy $b$ to $G(k,h)$
- $u(b|k,h) = E[u(\underbrace{O(b)}_{\text{lottery over terminal nodes induced by } b})|k,h] = E[u(b^{k,h})]$
- $u(\sigma|k,h) = E[u(\sigma)|k,h] = E[u(\sigma^{k,h})]$

Defn: $G$ is continuous at infinity iff $\forall \varepsilon > 0, \exists k$ s.t.

$$[h^k = \hat{h}^k] \Rightarrow |u_i(h) - u_i(\hat{h})| < \varepsilon$$

- Discounting is a sufficient condition for this

Suppose there are two behavioral strategies $b, \hat{b}$ s.t. $b(h^k) = \hat{b}(h^k)$ $\forall k \leq K$

- Given $\varepsilon > 0$, $\exists K > 0$ s.t. $|u(h) - u(\hat{h})| < \varepsilon$
- uniform in $h$.

Single-deviation principle:

Thm: Let G be a multistage game which is continuous at infinity. Then b is a SPE of G iff $\forall i, k, h,$

$$[\forall k' > k, h' : \underline{b}_i^{k',h'} = b_i^{k',h'}] \Rightarrow u_i(b|k,h) \geq u_i(\underline{b}_i, b_{-i}|k,h) \quad (*)$$

∘ Don't worry about what will happen in the future. Consider only one-period deviations

Pf: Suppose (*) holds but we are not in an SPE.

Then $\exists i, k^*, h^*$ s.t. $u_i(\underline{b}_i, b_i | k^*, h^*) > u_i(b|k^*, h^*)$

$|u_i(\underline{b}_i, b_{-i}|k^*, h^*) - u_i(b|k^*, h^*)| \equiv \varepsilon > 0$

Take $\frac{\varepsilon}{2}$. Define $b_i^0 = \begin{cases} \underline{b}_i & \text{if } k \leq K \\ b_i & \text{if } k > K \end{cases}$. There exists K

such that $|u_i(b_i^0, b_{-i}|k^*, h^*) - u_i(\underline{b}_i, b_{-i}|k^*, h^*)| < \frac{\varepsilon}{2}$.

Then $u_i(b_i^0, b_{-i}|k^*, h^*) > u_i(b|k^*, h^*)$ for some K where

Let $b_i^m = \begin{cases} \underline{b}_i & \text{if } k \leq K-m \\ b_i & \text{if } k > K-m \end{cases}$

Then, by (*), $\forall m, h^{K-m}$: $u_i(b_i^m, b_{-i}|K-m, h^{K-m})$

$\geq u_i(b_i^{m-1}, b_{-i}|K-m, h^{K-m})$ (ie no profitable one stage deviation)

Then $\forall m$, $u_i(b_i^m, b_{-i}|k^*, h^*) \geq u_i(b_i^{m-1}, b_{-i}|k^*, h^*)$

- $E[u(b^m)|k^*,h^*] = E[u(b^m) \mid K-m$ is not reached $|k^*,h^*]$
$+ E[u(b^m) \mid K-m$ is reached $|k^*,h^*]$

- $E[u(b^{m-1})|k^*,h^*] = E[u(b^m) \mid K-m$ is not reached $|k^*,h^*]$
$+ E[u(\underline{b}) \mid K-m$ is reached $|k^*,h^*]$

$\Rightarrow$ but we know that $u(b^m) > u(b^{m-1})$ if $K-m$ is reached. Then,

$E[u(b^m)|k^*,h^*] > E[u(b^{m-1})|k^*,h^*)]$

Thus, $u_i(b|k^*,h^*) = u_i(b_i^{K-k}, b_{-i}|k^*,h^*)$

$\geq u_i(b_i^0, b_{-i}|k^*,h^*) > u_i(b|k^*,h^*) \rightarrow\leftarrow$

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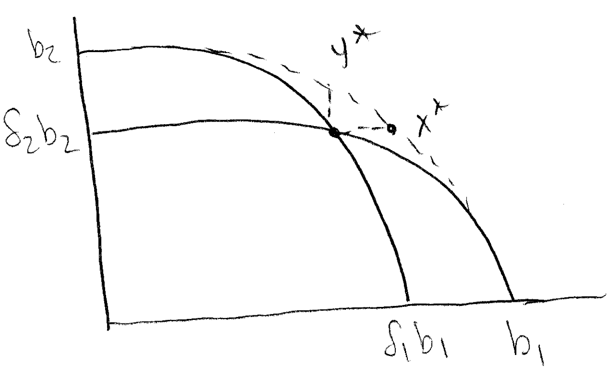
SPE in bargaining




Thm: The unique SPE is
- P1 always offers $x^*$
- P2 accepts iff $x_2 \geq x_2^*$
- P2 always offers $y^*$
- P1 accepts an offer $y$ iff $y_1 \geq y_1^*$

Pf: Use single deviation principle:
- spse player two was offered $x$ s.t. $x_2 \geq x_2^*$
  - accept. If reject, play according to SPE in the future. Get $y_2^*$ with payoff $\delta_2 y_2^*$, which

is equal to $x_2^*$. Since $x_2 \geq x_2^*$, this is not a profitable deviation.

- Spse player two is offered $x$ s.t. $x_2 < x_2^*$.
  - Reject and get $\delta y_2^* = x_2^* > x_2$, which is what 2 would get if accept. No profitable deviation.
- Suppose player 1 offers some $x \neq x^*$. If $x_2 > x_2^*$, it will be accepted, but by convexity of payoff set, $x_1 < x_1^*$. This is not a profitable deviation.
  - If $x_2 < x_2^*$, it will be rejected. Next period, will receive offer $y_1^*$ and get payoff $\delta y_1^* < y_1^* < x_1^*$. Not profitable.

<u>Iterated conditional dominance</u>

- $a_i^k(h)$ is conditionally dominated at $h$ iff each $b$ with $b_i(a_i^k(h)|h) > 0$ is strictly dominated by some $\underline{b}_i$: $\forall s_{-i}$,

$$u_i(\underline{b}_i, s_{-i}|k,h) > u_i(b_i, s_{-i}|k,h)$$

Iteratively eliminate all conditionally dominated actions

- This is a generalization of backward induction to any multi-stage game with observable actions.

Thm: In a game of perfect information, every SPE survives iterated conditional dominance.