Bayesian Games:

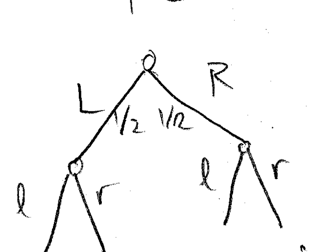
- We can replace p with $p_1, \ldots, p_n$, dropping the common prior assumption. $p \in \Delta(\Theta \times T)$

Normal form representation

- Given $\Gamma = (N, A, \Theta, T, u, p)$, the normal form game $G(\Gamma) = (N, S, U)$ is defined ex ante.
  - S_i: { functions $s_i : T_i \to A_i$ } (contingent plans)
    - $s_i : t_i \to s_i(t_i) \in A_i$
  - $U_i(s) = E_p[u_i(\theta, s_1(t_1), \ldots, s_n(t_n))]$
    - don't know your own type
  - This is a normal form game with complete information.

Interim: agent normal form: $AG(\Gamma) = (\underline{N}, S, U)$

- $\underline{N} = T_1 \cup \ldots \cup T_n$ (types are realized)
- $S_{t_i} = A_i$ for each $t_i \in T_i$
- $U_{t_i}(s) = E_p[u_i(\theta, s_1(t_1), \ldots, s_n(t_n)) \mid t_i]$
  - Each type is, in essence, a different player.



$G(\Gamma)$: $N = \{1\}$

$S = \{ll, lr, rl, rr\}$

$U(ll) = \frac{1}{2}$

$AG(\Gamma)$: $\underline{N} = \{L, R\}$

- $S_L = \{l, r\} = S_R$
- $S = S_L \times S_R$
- $U_L(l, l) = 1$

Bayesian Nash Equilibrium (this is an interim notion)

Defn: $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$ is a BNE iff $\forall i, t_i$,

$\sigma_i^*(a_i|t_i) > 0 \Rightarrow a_i \in \arg\max_{a_i} E_p[u_i(\theta, a_i, \sigma_{-i}^*(a_{-i}|t_{-i}))|t_i]$

- σ^* is a BNE of Γ iff $\sigma^*(\cdot|t_i), t_i \in T_i, i \in \underline{N}$ is a Nash equilibrium of $AG(\Gamma)$
- If σ^* is a BNE of of Γ, then σ^* is a NE of $G(\Gamma)$. If $p(t_i) > 0$ for each t_i, the converse is true.
  - There might be a BNE that is not a NE of $G(\Gamma)$, since some types might have zero probability.

Existence:

- Consider $\Gamma = (N, A, \Theta, T, u, p)$ with N, T finite

Thm: If
- each A_i is compact and convex
- each u_i is bdd, continuous in a, concave in a_i,

then Γ has a pure BNE.

Pf: $AG(\Gamma)$ has a pure NE.
- $\underline{N}$ is finite
- $S_{t_i} = A_i$ is convex and compact
- $U_{t_i}(s) = E_p[u_i(s_1(t_1), \dots, s_n(t_n))|t_i]$ is a weighted sum of continuous functions

- Suppose $s^m \to s$. Then we want to show that $\lim_{m\to\infty} U_{t_i}(s^m) = U_{t_i}(s)$
- $\lim_{m\to\infty} U_{t_i}(s^m) = \lim_{m\to\infty} E_p[u_i(s_1(t_1),\ldots,s_n(t_n))|t_i]$

$$\overset{T \text{ is finite}}{=} E_p[\lim_{m\to\infty} u_i(s_1(t_1),\ldots,s_n(t_n))|t_i]$$
$$= E_p[u_i(s_1(t_1),\ldots,S_n(t_n))|t_i]$$
$$= U_{t_i}(s)$$

- $U_{t_i}(s)$ is concave in a_i since u_i is concave.

Corollary: If A is finite, Γ has a (possibly mixed) Bayesian Nash equilibrium.

Upper-hemicontinuity

- A, T finite, Θ, X compact
- $u_i^x(\theta, a)$ continuous in (x, θ, a)
- $BNE(x)$ - Bayesian Nash Eq. of $\Gamma^x = (N, A, \Theta, T, u^x, p)$
- $BNE(\tilde{p})$ - BNE of $\Gamma^p = (N, A, \Theta, T, u, \tilde{p})$

Thm: BNE is upper-hemicontinuous

Pf: $BNE(x) = NE(AG(\Gamma^x))$. Follows by continuity of u_i

Thm: assume $p(t_i) > 0$ $\forall p \in P$, $\forall t_i \in T_i$, for compact $P \subseteq \Delta(\Theta \times T)$. $BNE(p)$ is uhc in P.

- $U_i(s; p) = E_p[u_i(\theta, s_1(t_1), \ldots, s_n(t_n)) | t_i]$ is continuous; $BNE(p) = NE(G(\underbrace{N, A, \theta, T, u, p}_{\Gamma}))$

Extensive form:

- N - players
- a tree
  - initial node, terminal nodes
  - directed, connected, acyclic graph with a unique path from the initial node to each terminal node
- payoffs for each player are defined at each terminal node
- Information partition (over non-terminal nodes)
  - If two nodes are in the same information set, the same actions must be available.
- Player map is a function from histories to the set of players. (including nature)
- perfect recall: player never forgets what he has known or done.

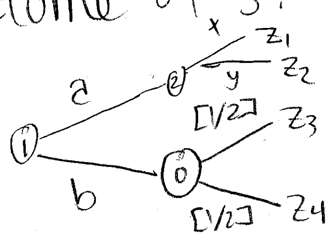
a tree is a directed graph $(X, <)$ with

- Initial node φ
- $<$ transitive: $(x < z, y < z) \Rightarrow x < z$
- asymmetric $x < y \Rightarrow \neg(y < x)$
- arborescence: $(x < z, y < z) \Rightarrow (x < y \text{ or } y < x)$

- Terminal nodes: $Z = \{z \mid \nexists x \text{ with } z < x\}$
- Utility $u_i : Z \to \mathbb{R}$
- $A(x)$ = available actions $x \in X \setminus Z$
- Information partition: H (of $X \setminus Z$)
    - $x, y \in h \in H \Rightarrow A(x) = A(y) \equiv A(h)$
- Player map: $i : H \to N \cup \{0\}$; $i(h)$ moves at h
    - If $i(h) = 0$, a probability dist on $A(h)$
    - $h_i \in H_i = \{h \mid i(h) = i\}$

Normal and reduced normal forms:

- Strategy set: $S_i = \prod_{h_i \in H_i} A(h_i)$, $s_i : h_i \longmapsto s_i(h_i) \in A_i(h_i)$
    - e.g. a strategy is a complete contingent plan.
- Outcome of s: $O(s) \in \Delta(Z)$

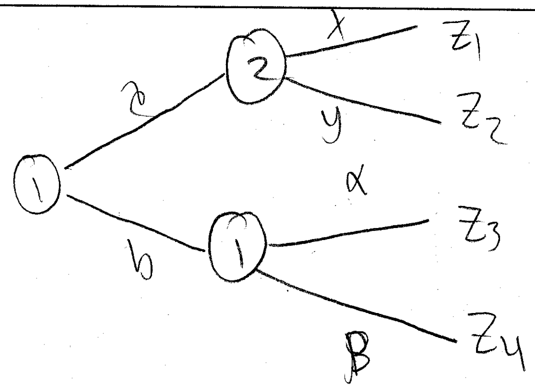


$O(a, x) = z_1$

$O(b, x) = \begin{cases} z_3 & \text{w/prob } 1/2 \\ z_4 & \text{w/prob } 1/2 \end{cases}$

- $u_i(s) \equiv u_i(O(s))$
- Normal form (N, S, u).

Consider the following extensive form game:



1 \ 2	x	y
$a\alpha$	z_1	z_2
$a\beta$	z_1	z_2
$b\alpha$	z_3	z_3
$b\beta$	z_4	z_4

- Here, $a\alpha$, $a\beta$ are equivalent.
- s_i, s_i' are <u>equivalent</u> iff $\forall s_{-i}: O(s_i, s_{-i}) = O(s_i', s_{-i})$
- $S_i^R = \{$a strategy from each equivalent class$\}$

$\Rightarrow$

1 \ 2	x	y
a	z_1	z_2
$b\alpha$	z_3	z_3
$b\beta$	z_4	z_4

- Reduced normal form: (N, S^R, u)