Existence and continuity of NE

Suppose there is some x, and that the modeler assumes that $x=\bar{x}$. Let $F(\bar{x})$ be the solution to the model. We want continuity of F. To see this, assume $x=\bar{x}-\varepsilon$ is the true value. Then if F is continuous, $\|F(\bar{x})-F(x)\| \leq \delta$ for δ sufficiently small if ε is sufficiently small.

e.g. Let $x \in B_\delta(\bar{x}) = \{x \mid d(x,\bar{x}) < \delta\}$

$$y \in B_\varepsilon(\bar{y}) = \{y \mid d(y,\bar{y}) < \varepsilon\}$$

$$B_\varepsilon(F(\bar{x})) = \bigcup_{\bar{y}\in F(\bar{x})} B_\varepsilon(\bar{y})$$

If $F(x) \subseteq B_\varepsilon(F(\bar{x}))$, you are fine.

Want: $\forall \varepsilon>0, \exists \delta>0$ s.t. $\forall x \in B_\delta(\bar{x}), F(x) \subseteq B_\varepsilon(F(\bar{x}))$

- This is called upper hemi-continuity.

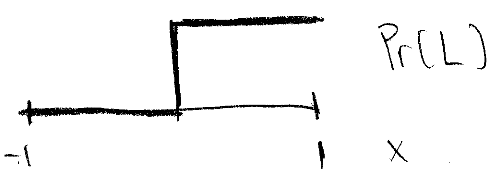
Thm: For any correspondence $F: X \to 2^Y$, where X is compact and Y bounded, F is upper-hemicontinuous iff F has a closed graph:

$[x_m \to x$ and $y_m \to y$ s.t. $y_m \in F(x_m)] \Rightarrow y \in F(x)$

L R , $x \in [-1, 1]$

x 0

- suppose a guy has a choice b/t going left and right. If he goes L, he receives x. If he goes right he receives zero. His BR correspondence is:



<u>Thm:</u> Assume $f: X \times Z \to Y$ is continuous and X, Y, Z are compact. Let $F(x) = \underset{z \in Z}{\operatorname{argmax}} f(x, z)$

Then F is nonempty, compact-valued, and upper-hemicontinuous.

<u>Kakutani:</u> Let $\emptyset \neq X$ be compact, convex and let $F: X \to 2^X$ be non-empty, convex-val. correspondence with closed graph. Then $\exists\, x \in X$ s.t. $x \in F(x)$

<u>Thm:</u> Let S_i be convex, compact subset of Euclidean space and u_i be continuous in S and quasi-concave in S_i. Then, $\exists$ a NE $s \in S$.

<u>Corollary:</u> Every finite game has a possibly mixed NE σ^*,

Existence theorem:

- Let $F: S \to 2^S$ be the BR correspondence: $F_i(s) = B_i(s_{-i})$
- By the maximum thm, F is nonempty and has a closed graph
- By quasiconcavity, F is convex valued
- $0 \neq X$, convex, compact was assumed
- By Kakutani, $\exists s^*$ s.t. $s^* \in F(s^*)$

$\Rightarrow s^*$ is a NE $\square$

Upper-hemicontinuity:

- assume X, S be compact metric spaces
- Spse $u^x(s)$ is continuous in $x \in X$ and $s \in S$.
- Let $NE(x)$ be the set of NE of (N, S, u^x)
- Let $PNE(x)$ be the pure strategy equilibria of (N, S, u^x)

Thm: NE, PNE are upperhemicontinuous.

Corollary: If S is finite, NE is nonempty, compact valued, and upper-hemicontinuous

	L	R
U	2,2	0,0
D	0,0	x,x

$x \in [-1, 1]$

$2p = x - xp$

$p(2+x) = x$

$p = \frac{x}{2+x}$

$$NE(x) = \begin{cases} (U,L) & x < 0 \\ (U,L), (D,R), & x = 0 \\ (U,L), (D,R), \left(\frac{x}{2+x}, \frac{2}{2+x}, \frac{x}{2+x}, \frac{2}{2+x}\right) & x > 0 \end{cases}$$

Pf: $\Delta(S_i)$ is compact and $\underbrace{U^x(\sigma)}_{=\sum_{s\in S}\sigma(s)u^x(s)}$ is continuous in (x, σ)

- Suppose $x_m \to x$, $\sigma^m \in NE(x_m)$, $\sigma^m \to \sigma \notin NE(x)$ to get a contradiction.
- $\exists$ i s.t. for some s_i, $U^x(s_i, \sigma_{-i}) > U^x(\sigma)$ $\Rightarrow U_i^{x^m}(s_i, \sigma_{-i}^m) > U^{x^m}(\sigma^m)$ for some m by convergence. Thus σ^m is not a NE.

Therefore, $NE(x)$ is upper-hemicontinuous.

Bayesian Games

Defn: a Bayesian Game is a list (N, A, Θ, T, u, p).

- N - set of players
- $A = A_1 \times \dots \times A_n$ - set of actions
- Θ - payoff relevant parameter
- $T = T_1 \times \dots \times T_n$ - type space
- $u = (u_1, \dots, u_n)$; $u_i: \Theta \times A \to \mathbb{R}$ is i's vNM u-fcn
- $p \in \Delta(\Theta \times T)$ is a common prior
  - $(\theta, t_1, \dots, t_n) \sim p$
- This structure is common knowledge
- $p(\theta | t_i)$ - i's first-order beliefs.