This class is more applied than in the past.

Normal-form games

Defn: a *normal form game* is a triple (N, S, u) s.t.

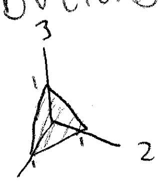
- $N = \{1, \ldots, n\}$ is a set of players
- $S = S_1 \times \ldots \times S_n$ is a strategy set
- $u_i : S \to \mathbb{R}$ is a vNM utility function.

We assume that the structure of the game is common knowledge.

a normal form game is *finite* if N and S are finite.

Let $\Delta(X)$ be the set of probability distributions on X.

- Suppose $X = \{1, 2, 3\} \Rightarrow \Delta(X)$ looks like:



Given S_i, $\Delta(S_i)$ is the set of mixed strategies for player i.

an *independent strategy profile* is: $\sigma = \sigma_1 \times \ldots \times \sigma_n \in \Delta(S_1) \times \ldots \times \Delta(S_n)$

a *correlated strategy profile* is: $\sigma \in \Delta(S)$.

e.g.

I \ II	L	R
U	1/3	1/3
D	0	1/3

Not independent because the probabilities are not the product of the marginal probabilities.

Let $\Delta(S_{-i})$ be the possible conjectures of player i. (beliefs about the other players' strategies)

- a player may believe that other players' strategies are correlated.

Expected payoffs: $u_i(\sigma) = E_\sigma[u_i] = \sum_{s\in S} \sigma(s) u_i(s)$

Defn: Player i is rational if i maximizes his payoff given his beliefs

Defn: s_i^* is a best reply to a belief σ_{-i} iff

$\forall s_i \in S_i: u_i(s_i^*, \sigma_{-i}) \geq u_i(s_i, \sigma_{-i})$

$B_i(\sigma_{-i})$ is the set of best replies to σ_{-i}

σ_i strictly (weakly) dominates s_i iff

$\forall s_{-i} \in S_{-i}: u_i(\sigma_i, s_{-i}) > (\geq)\ u_i(s_i, s_{-i})$

Thm: In a finite game, s_i^* is not a best reply to any conjecture σ_{-i} iff s_i^* is strictly dominated

Pf: Let $S_{-i} = \{s_{-i}^1, \ldots, s_{-i}^m\}$ be the strategy profiles about which i is uncertain.

- Let $u_i(s_i, \cdot) = \{u_i(s_i, s_{-i}^1), \ldots, u_i(s_i, s_{-i}^m)\}$
- Define $U = \{u_i(s_i, \cdot) : s_i \in S_i\}$
- Let $Conv(U)$ be the convex hull of this set
  - $Conv(U) = \{u_i(\sigma_i, \cdot) : \sigma_i \in \Delta(S_i)\}$

($\Rightarrow$) assume $s_i^* \in B_i(\sigma_{-i})$

- Then $\forall s_i$, $u_i(s_i^*, \sigma_{-i}) \geq u_i(s_i, \sigma_{-i})$

Thus, $\forall \sigma_i \in \Delta(S_i)$, $u_i(s_i^*, \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i})$.

$$= \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, \sigma_{-i})$$

$\Rightarrow$ No σ_i strictly dominates s_i^*.

Recall (Sep. Hyperplane Thm): Let C, D be nonempty disjoint subsets of $\mathbb{R}^m$ which are convex. Let C be closed. Then $\exists r \in \mathbb{R}^m \setminus \{0\}$ s.t. $\forall x \in cl(D), \forall y \in C, r \cdot x \geq r \cdot y$

($\Leftarrow$) Define $D = \{x \in \mathbb{R}^m \mid x_k > u_i(s_i^*, s_{-i}^k) \ \forall k\}$

Assume s_i^* is not strictly dominated.

Then $Conv(U) \cap D = \emptyset$.

By the SHT, $\exists r$ s.t. $\forall x \in D, \forall \sigma_i$,

$r \cdot x \geq r \cdot u_i(\sigma_{-i}, \cdot)$

In particular, $r \cdot u_i(s_i^*, \cdot) \geq r \cdot u_i(\sigma_i, \cdot)$

$$\Rightarrow \underbrace{\frac{r}{\sum r_k}}_{\text{this is a belief}} u_i(s_i^*, \cdot) \geq \frac{r}{\sum r_k} u_i(\sigma_i, \cdot)$$

$$\Rightarrow \sigma_{-i} \cdot u_i(s_i^*, \cdot) \geq \sigma_{-i} \cdot u_i(\sigma_i, \cdot)$$

$$\Rightarrow u_i(s_i^*, \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i})$$

Iterated Strict Dominance and Rationalizability

- Let $S^0 = S$

- Define $S_i^1 = B_i(\Delta(S_{-i}^0))$, $S^1 = S_1^1 \times \dots \times S_n^1$
  - ie we throw out strictly dominated strategies.
- $S_i^k = B_i(\Delta(S_{-i}^{k-1}))$

- Rationalizable (correlated) strategies:

$$S_i^\infty = \bigcap_{k=0}^{\infty} S_i^k$$

- Independent rationalizability: $s_i \in S_i^k$ iff

$$s_i \in B_i\left(\prod_{j \neq i} \sigma_j\right), \text{ where } \sigma_j \in \Delta(S_j^{k-1}) \ \forall j$$

- σ_i is rationalizable iff $\sigma_i \in B_i(\Delta(S_{-i}^\infty))$

<u>Thm:</u> S^∞ is the largest set $Z_1 \times \dots \times Z_n$ s.t.

$Z_i \subseteq B_i(\Delta(Z_{-i})) \ \forall i$.

(ie $\forall z_i \in Z_i$, $\exists \sigma_{-i} \in \Delta(Z_{-i})$ s.t. $z_i \in B_i(\sigma_{-i})$)

(ie. S^∞ is closed under best reply.)

If the game and rationality are common knowledge, then each player must play a rationalizable strategy.

Further, for all rationalizable strategy profiles, there is a game for which this is the result under common knowledge.

Under learning, the mass of players playing non-rationalizable strategies goes to zero.

Nash Equilibrium

TFAE:

1. $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is a NE
2. $\forall i$, $\sigma_i^* \in B_i(\sigma_{-i}^*)$ where B_i contains mixed best replies.
3. $\forall i$, $\forall s_i \in S_i$, $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*)$
4. $\forall i$, $\text{supp}(\sigma_i^*) \subseteq B_i(\sigma_{-i}^*)$

aumann, Brandenburger: In a 2 player game, if game, rationality, and conjectures are mutually known, then the conjectures constitute a Nash Equilibrium.

- For $n > 2$, this theorem need not hold. We need common priors and common knowledge.