Optimal auctions (private iid values)

- Risk neutrality for everybody
- continuum of types
- $\theta_i \sim f_i(\theta_i), F_i(\theta_i)$ (we are temporarily dropping symmetry assumption)
  pdf/cdf depend on i
- assume independent private values

Consider a multi-unit auction:

$$(F): \sum_{i=1}^{N} q_i(\underbrace{\theta}) \leq q_0$$

vector of announced types: $\theta = (\theta_1, \dots, \theta_N)$

- $u_i(q, T) = \theta_i V(q) - T$

What is the set of implementable mechanisms?

- must satisfy (IR) and (IC) in expectation as well as feasibility. (over others' types)

$$(IC): \pi_i(\theta_i, \theta_i) = \max_{\hat{\theta}_i} \pi_i(\underbrace{\hat{\theta}_i}_{\text{announced type}}, \underbrace{\theta_i}_{\text{true type}})$$

where $\pi_i(\hat{\theta}_i, \theta_i) = \underbrace{E_{\theta_{-i}}}_{\text{true types}}[\theta_i V(q_i(\hat{\theta}_i, \underbrace{\theta_{-i}})) - T_i(\hat{\theta}_i, \theta_{-i})]$

looking for BNE with truth-telling

Call $T_i(\hat{\theta}_i) \equiv E_{\theta_{-i}}[T(\hat{\theta}_i, \theta_{-i})]$

It is easy to show that $(IC) \Leftrightarrow (LIC) + (M)$,

where: $(M) \Leftrightarrow E_{\theta_{-i}}[q_i(\theta_i, \theta_{-i})]$ increasing in θ_i.

This follows from Spence-Mirrlees, since:

$$\frac{\partial}{\partial\theta}\left[\frac{\partial u_i/\partial q}{\partial u_i/\partial(-T)}\right] = v'(q) > 0.$$

(LIC): Thus, $\pi_i(\theta_i,\theta_i) = \pi_i(\underline{\theta},\underline{\theta}) + E_{\theta_{-i}}\left[\int_{\underline{\theta}}^{\theta_i} v(q_i(x,\theta_{-i}))dx\right]$

where $\underline{\theta}$ is the lower bound on the support of the distributions.

The participation constraints are:

- $\pi_i(\theta_i,\theta_i) \geq 0 \quad \forall i, \forall \theta_i \qquad (IR_{\theta_i})$

If $(IR_{\underline{\theta}})$ holds, then all (IR_{θ}) hold.

The (LIC) gives us:

$$T_i(\theta_i) = E_{\theta_{-i}}\left[\theta_i v(q_i(\theta)) - \pi_i(\underline{\theta},\underline{\theta}) - \int_{\underline{\theta}}^{\theta_i} v(q_i(x,\theta_{-i}))dx\right]$$

Then, $T_i^e = \int_{\underline{\theta}}^{\bar{\theta}} E_{\theta_i}\left[\theta_i v(q_i(\theta)) - \pi_i(\underline{\theta},\underline{\theta}) - \int_{\underline{\theta}}^{\theta_i} v(q_i(x,\theta_{-i}))dx\right]\underbrace{dF_i(\theta_i)}_{=f_i(\theta_i)d\theta_i}$

Total expected revenue is: $T^e = \sum_{i=1}^{N} T_i^e$

- depends on: rent of lowest type and allocation of the goods.
- This is the revenue equivalence theorem.

Revenue Equivalence Theorem

In the continuum case (with private, independent values and risk neutrality), seller revenue is uniquely determined by $\pi_i(\underline{\theta},\underline{\theta})$ and $q_i(\theta)$.

Remark: If standard auctions all have the same $\pi_i(\underline{\theta},\underline{\theta})$ and $q_i(\theta)$, they will have the same expected revenue.

- Vickrey: people bid their own valuation and winner pays 2nd highest value
  - ex-post efficient (and $\pi_i(\underline{\theta},\underline{\theta})=0$)

Can use integration by parts to get

$$T_i^e = \int_{\underline{\theta}}^{\bar{\theta}} E_{\theta_{-i}}\left[\theta_i V(q_i(\theta)) - \pi_i(\underline{\theta},\underline{\theta}) - \frac{V(q_i(\theta_i,\theta_{-i}))}{h_i(\theta_i)}\right] f_i(\theta_i)\,d\theta_i,$$

where $h_i(\theta_i) = \dfrac{f_i(\theta_i)}{1-F_i(\theta_i)}$

Optimal auctions: $(\Rightarrow \pi_i(\underline{\theta},\underline{\theta})=0)$

assume:

- symmetry: $(f_i(\cdot)=f(\cdot))$
- $\theta_i v(q) = \begin{cases} \theta_i q & \text{if } q \le 1 \\ \theta_i & \text{if } q > 1 \end{cases}$
- care only about one unit of consumption.

- Here, we have:

$$\max_{q_i(\theta)} E_\theta\left[\sum_{i=1}^{N}\left\{\theta_i \min\{q_i,1\} - \frac{\min\{q_i,1\}}{h(\theta_i)}\right\}\right]$$

$$\text{s.t.} \quad \underbrace{\sum_{i=1}^{N} q_i(\theta) \le q_0}_{(F)} \ \forall\theta, \text{ and } \ (M)$$

$$\Leftrightarrow \max_{q_i(\theta)} E_\theta\left[\sum_{i=1}^{N} J(\theta_i)\min\{q_i,1\}\right], \quad J(\theta_i) = \left[\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}\right]$$

s.t. (F) and (M).

Result: Sell to highest $J(\theta_i)$'s provided they have positive $J(\theta_i)$'s. (and $\sum q_i(\theta) \le q_0$)

- will have (M) if $J(\theta_i) \uparrow$ in θ_i
  - sufficient: $\frac{f_i}{1-F_i} \uparrow$ w/ θ_i

The optimal auction is ex-post efficient, possibly with a floor.

$\Leftrightarrow$ Vickrey, with a floor! (reserve price)

interestingly

- floor: $J(\tilde{\theta}) = 0$ $\quad \hat{\theta}$ is a floor $\Rightarrow$ $\theta < \hat{\theta} \Rightarrow$ do not participate

First-price auction

Look for a symmetric Bayesian equilibrium, where bids increase strictly with type $\Leftrightarrow$ ex-post efficiency

maximization problem for each bidder i:

$$\max_{b_i} \ (\theta_i - b_i)\underbrace{g_i(b_i)}_{\text{probability of winning}}$$

- assume two bidders
- $\theta_i \sim U[0,1]$ iid.

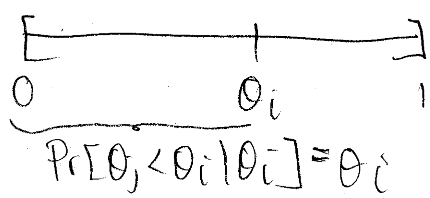
<u>FOCs:</u>

$$(b_i):\ (\theta_i - b_i)g'(b_i) - g(b_i) = 0$$

Symmetry implies $g_i(b_i) = g(b_i) = \theta_i$, since

$$Pr[\text{win}] = Pr[\underbrace{\theta_i > \theta_j}\,|\,\theta_i] = \theta_i$$

$$= Pr[g(\theta_i) > g(\theta_j)\,|\,\theta_i]$$



Thus, the FOCs become:

$$[g(b_i) - b_i]g'(b_i) = g(b_i) \Rightarrow \underbrace{g(b_i)}_{=\theta_i} = 2b_i$$

$$\Rightarrow b_i^* = \frac{\theta_i}{2}$$

If values are not independent, the seller can do better.

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