The core

Connection to prices. Critical interpretation

2 cases:

$I = \{1, \ldots, n\}$ $\quad \mathcal{I} \equiv \mathcal{P}(I)$, $S \in \mathcal{I}$

$I = [0,1]$ $\quad \mathcal{I} \equiv \mathcal{B}(I)$, $S \in \mathcal{I}$

Game in characteristic function form

$V : \mathcal{I} \to \mathbb{R}_+$ set function

$V(S)$ - worth of coalition S.

$V(\emptyset) = 0$ by convention.

Think of the game as (V, I)

Defn: The core of V is $C(V) = \{q = (q_i)_{i\in I} : \sum_{i\in I} q_i = V(I), \sum_{i \in S} q_i \geq V(S)\}$

Linear programming problem: Let $1_S = (0, \ldots, 0, \overbrace{1, \ldots, 1}^{S}, 0, \ldots, 0)$

$$(P) \quad h(1_I) = \max\left\{\sum_S V(S)x(S) : \sum_S 1_S x(S) = 1_I, \; x(S) \geq 0\right\} \underset{\substack{\uparrow \\ \text{trivially}}}{\geq} V(I)$$

(set $x(I) = 1$, $x(S) = 0$ $\forall S \neq I$)

2^n activities.

h is positively homogeneous and superadditive.

$$(D) \quad \min_{(q_i)_{i\in I}} \left\{\sum_{i \in I} q_i : \sum_{i\in S} q_i \geq V(S) \;\; \forall S \in \mathcal{I}, \; q_i \geq 0 \;\forall i \in I\right\}$$

(x, q) are optimal for (P) and (D) gives us that

$$\sum_{i\in I} q_i = \sum_S V(S)x(S) = h(1_I)$$

also, $q \in \partial h(1_I) \Rightarrow \sum_{i \in I} q_i = h(1_I)$, $\sum_{i \in I} q_i r_i \geq h(r)$, $r = (r_1, \ldots, r_n)$

$$\pi = h(r) - q \cdot r \leq h(1_I) - q \cdot 1_I \quad \text{since } q \in \partial h(1_I)$$

Typically, we assume that if $S \cap T = \emptyset$, $V(S \cup T) \geq V(S) + V(T)$

An integral optimal solution is a partition of 1_I

Suppose the game is not superadditive. Then take the superadditive cover in which if $V(S \cup T) < V(S) + V(T)$, set $V(S \cup T) = \max\{V(S), V(T)\}$

$$\hat{V}(S) = \max\{ \sum_k V(S_k) : \bigcup_k S_k = S,\ S_k \cap S_j = \emptyset \}$$

If our game is superadditive and has an integral optimal solution, then $h(1_I) = \max\{ \sum_S V(S)x(S) : \sum 1_S x(S) = 1_I,\ x(S) \geq 0 \}$
$= V(I)$

$\Rightarrow$ The game has a nonempty core.

$\frac{h(k 1_I)}{k} \longrightarrow V(I)$, $k \to \infty$ asymptotic replica invariance

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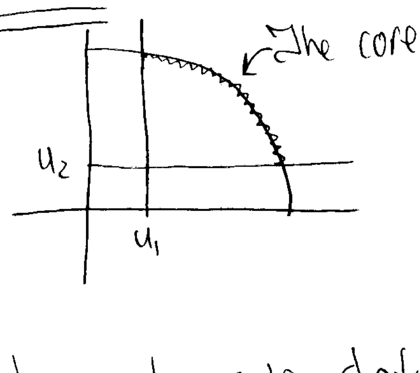
Let $I = \{1,2,3\}$
$V(1,2,3) = 1$
$V(i,j) = 2/3 = \ell$
$V(i) = 0$

Can we find a superadditive game with no integral solutions?
Superadditive game.
If $\ell > 2/3$, cannot find integrally optimal solution.

Connection between the core and pricing equilibrium in terms of commodities.



Boundary of $V(I)$

$$\max\{ \sum_{i \in I} \lambda_i u_i : \lambda_i \geq 0,\ \sum_{i \in I} \lambda_i = 1,\ (u_1, \ldots, u_n) \in V(I) \}$$

└ can trace out all the points on the boundary by varying the weights.

How do we define the core in the ordinal world?

Core: $(u_1, \dots, u_n) \in \text{Boundary}(V(I))$
$(u_i)_{i\in S} \notin \text{Int}\, V(S) \quad \forall S \neq I$

$$\boxplus_{i\in S} V_i(0) + \boxplus_{i\in T} V_i(0) \le \boxplus_{i\in S\cup T} V_i(0) \quad \text{if } S\cap T = \emptyset.$$

Let $I = \{1, \dots, n\}$, $V: \mathcal{A} \to \mathbb{R}_+$, $\mathcal{A} \subseteq P(I)$
There may be many economic models which map into V.

Ch. fcn
$V: \mathcal{A} \to \mathbb{R}_+$

Economic models

Need to reproduce $V(123)=1$, $V(ij)$, $V(i)=0$. Only need three numbers: $V(12)$, $V(13)$, $V(23)$.
The concept of the core says that it does not matter which economic model is giving us this V function.
For instance: assignment model
$I = B \cup S \qquad |S| = \ell$
$V(T) = 0$ if $T \subset B$ or $T \subset S$
$V(T) = \max\{ \sum_{b\in B_T, s\in S_T} V(b,s) : B_T = T\cap B,\ S_T = T\cap S\}$?

$V_b, V_s : \mathbb{R}^\ell \to \mathbb{R} \qquad V_b(z) = b_c$ if $z = (0, \dots, 0, \overset{c}{1}, 0, \dots, 0)$ $\forall c = 1, \dots, \ell$
$V_b(0) = 0$
$V_s(z) = -s_c$ if $z = (0, \dots, 0, \overset{c}{-1}, 0, \dots, 0)$ $\forall c = 1, \dots, \ell$ when $c = s$ (ie each seller has its own commodity)
$V_s(0) = 0$

Then, $V(b,s) = \max \{ v_b(z) + v_s(-z) \}$

From this, we can construct $V(T)$ $\forall T \subset I$ by just taking maximum gains from trade for the group T.

This becomes a linear programming problem, where 1_I is a resource constraint, as is $\sum z_i x(i) = 0 \in \mathbb{R}^\ell$ (ie we have a resource constraint.)

$1_I \mapsto q$ dual prices for people

$0 \mapsto p$ dual prices for commodity.

We can alternatively allow each seller to supply one unit of any commodity, and we can get the same V.

alternatively, we can create a matrix game and interpret the prices for the 0 constraint as contractual prices. We can still obtain the same V.

If the core is <u>really</u> the thing to look at, then looking at q is the same as looking at (p, P). (core equivalence theorem). But the core can be bigger than the set of Walrasian equilibria.

$q_i \longleftrightarrow v_i^*(p, P)$

$\longleftrightarrow$ something else (sometimes)