Econ 203C: Systems Models

Problem Set 3

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Question 1:
This question is based on a statement made in Lecture Note 5. Consider the IV estimator of the form

$$\hat{\beta}_{IV}\left(\hat{\Pi}\right) = \left(\hat{\Pi}' Z' X\right)^{-1} \hat{\Pi}' Z' y$$

(1) Suppose that $\hat{\Pi} \xrightarrow{p} \Pi_0$ as $n \to \infty$. Provide the asymptotic distribution for $\hat{\beta}_{IV}\left(\hat{\Pi}\right)$ and the asymptotic covariance matrix, say Λ_0.

Solution: We can rewrite the estimator as

$$\begin{aligned} \hat{\beta}_{IV}\left(\hat{\Pi}\right) &= \left(\hat{\Pi}' Z' X\right)^{-1} \hat{\Pi}' Z' \left(X\beta + \varepsilon\right) \\ &= \beta + \left(\hat{\Pi}' Z' X\right)^{-1} \hat{\Pi}' Z' \varepsilon \\ &= \beta + \left(\hat{\Pi}' \frac{1}{n} \sum_{i=1}^{n} z_i x_i'\right)^{-1} \hat{\Pi}' \frac{1}{n} \sum_{i=1}^{n} z_i \varepsilon_i \end{aligned}$$

Or

$$\sqrt{n}\left(\hat{\beta}_{IV}\left(\hat{\Pi}\right) - \beta\right) = \left(\hat{\Pi}' \frac{1}{n} \sum_{i=1}^{n} z_i x_i'\right)^{-1} \hat{\Pi}' \sqrt{n}\left(\frac{1}{n} \sum_{i=1}^{n} z_i \varepsilon_i\right)$$

Assuming z_i, x_i, ε_i are i.i.d, we have that, by the weak law of large numbers, assuming the proper moment conditions,

$$\frac{1}{n} \sum_{i=1}^{n} z_i x_i' \xrightarrow{p} E\left[z_i x_i'\right] \equiv \Sigma_{zx}$$

By Slutsky's theorem and the Mann-Wald theorem (combined and known as Jin's theorem),

$$\left(\hat{\Pi}' \frac{1}{n} \sum_{i=1}^{n} z_i x_i'\right)^{-1} \hat{\Pi}' \xrightarrow{p} \left(\Pi_0' \Sigma_{zx}\right)^{-1} \Pi_0'$$

By the central limit theorem for i.i.d. data, assuming the proper moment conditions, we have that

$$\sqrt{n}\left(\frac{1}{n} \sum_{i=1}^{n} z_i \varepsilon_i - \underbrace{E\left[z_i \varepsilon_i\right]}_{=0}\right) \xrightarrow{d} N\left(0, V_0\right)$$

Where $V_0 = E\left[z_i \varepsilon_i \left(z_i \varepsilon_i\right)'\right] = E\left[z_i z_i' \varepsilon_i^2\right]$. Once again invoking Slutsky's theorem, we have

$$\sqrt{n}\left(\hat{\beta}_{IV}\left(\hat{\Pi}\right) - \beta\right) \xrightarrow{d} N\left(0, \Lambda_0\right)$$

Where

$$\Lambda_0 = \left(\Pi_0' \Sigma_{zx}\right)^{-1} \Pi_0' V_0 \Pi_0 \left(\Sigma_{zx}' \Pi_0\right)^{-1}$$

(2) Suppose now that $\hat{\Pi} \xrightarrow{p} V_0^{-1} \Sigma_{zx}$, where V_0 and Σ_{zx} are defined in the lecture note. Provide the asymptotic distribution for $\hat{\beta}_{IV}\left(\hat{\Pi}\right)$ and the asymptotic covariance matrix, say Λ^*.

Solution: Here, we have $\Pi_0 = V_0^{-1}\Sigma_{zx}$. By the argument from part (1),

$$\sqrt{n}\left(\hat{\beta}_{IV}\left(\hat{\Pi}\right) - \beta\right) \xrightarrow{d} N\left(0, \Lambda^*\right)$$

Where

$$\begin{aligned}
\Lambda^* &= \left(\Sigma'_{zx}\left(V_0^{-1}\right)'\Sigma_{zx}\right)^{-1}\Sigma'_{zx}\left(V_0^{-1}\right)'V_0V_0^{-1}\Sigma_{zx}\left(\Sigma'_{zx}V_0^{-1}\Sigma_{zx}\right)^{-1} \\
&= \left(\Sigma'_{zx}V_0^{-1}\Sigma_{zx}\right)^{-1}
\end{aligned}$$

(3) Show that the asymptotic covariance matrix in (1) is at least as large as the asymptotic covariance matrix in (2) in the matrix sense, that is, show that $\Lambda_0 - \Lambda^*$ is a positive definite matrix.

Solution: Recall that if A, B are positive semi-definite, we have that $A - B$ is positive semi-definite if and only if $B^{-1} - A^{-1}$ is positive semi-definite. Let $A = \Lambda_0$ and $B = \Lambda^*$.

$$\begin{aligned}
\left(\Lambda^*\right)^{-1} - \Lambda_0^{-1} &= \Sigma'_{zx}V_0^{-1}\Sigma_{zx} - \Sigma'_{zx}\Pi_0\left(\Pi_0'V_0\Pi_0\right)^{-1}\Pi_0'\Sigma_{zx} \\
&= \Sigma'_{zx}\left(V_0^{-1} - \Pi_0\left(\Pi_0'V_0\Pi_0\right)^{-1}\Pi_0\right)\Sigma_{zx} \\
&= \Sigma'_{zx}V_0^{-1/2}\left(I - V_0^{1/2}\Pi_0\left(\Pi_0'V_0\Pi_0\right)^{-1}\Pi_0\left(V_0^{1/2}\right)'\right)\left(V_0^{-1/2}\right)'\Sigma_{zx} \\
&= C\left(I - D\left(D'D\right)D'\right)C' \\
&= CM_{V_0^{1/2}\Pi_0}C' \\
&= CM_{V_0^{1/2}\Pi_0}M'_{V_0^{1/2}\Pi_0}C' \\
&= \left(M'_{V_0^{1/2}\Pi_0}C'\right)'M'_{V_0^{1/2}\Pi_0}C' \text{ by idempotence of } M_{V_0^{1/2}\Pi_0} \\
&= W'W
\end{aligned}$$

Where $C = \Sigma'_{zx}V_0^{-1/2}$ and $D = V_0^{1/2}\Pi_0$. Thus, since $W'W$ is a positive semi-definite matrix, we have that $\left(\Lambda^*\right)^{-1} - \Lambda_0^{-1}$ is positive semi-definite, and finally, we can conclude that $\Lambda_0 \geq \Lambda^*$ in the matrix sense.

Question 2:

Consider the SUR model given by

$$y_{ij} = x_{ij}'\beta_j + \varepsilon_{ij}, \quad i = 1, \ldots, n; \; j = 1, 2, 3,$$

with

$$\varepsilon_i \overset{i.i.d.}{\sim} D(0, \Sigma),$$

where $\varepsilon_i = (\varepsilon_{i1}, \varepsilon_{i2}, \varepsilon_{i3})'$. Also, assume that $E[x_{ij}\varepsilon_i] = 0$ for $j = 1, 2, 3$. In the EXCEL file ps3q2.xls, you are provided with the data for this problem set. There are 13 columns and 500 rows (i.e., observations) in this file. The first three columns contain the data on y_1, y_2, and y_3, respectively. The rest of the columns contain the data on $x_1, \ldots, x_{10}$. In addition there is a matlab file called ps3q2.mat that contains the same data in $y1, y2, y3$, and X.

Suppose now that the first equation has only x_1, x_2, x_3, and x_4 as explanatory variables, the second equation has x_5, x_6, x_7, and x_8 as explanatory variables, while the third equation has x_1, x_2, x_5, x_6, x_9, and x_{10} as explanatory variables.

(1) Provide the OLS estimate for the parameter vectors β_j, $j = 1, 2, 3$.

Solution: Let

$$\begin{aligned} X_{i1}' &= \begin{bmatrix} x_{i1} & x_{i2} & x_{i3} & x_{i4} \end{bmatrix} \\ X_{i2}' &= \begin{bmatrix} x_{i5} & x_{i6} & x_{i7} & x_{i8} \end{bmatrix} \\ X_{i3}' &= \begin{bmatrix} x_{i1} & x_{i2} & x_{i5} & x_{i6} & x_{i9} & x_{i10} \end{bmatrix} \end{aligned}$$

And stacking the variables,

$$X_1 = \begin{bmatrix} X_{1,1}' \\ \vdots \\ X_{500,1}' \end{bmatrix}, X_2 = \begin{bmatrix} X_{1,2}' \\ \vdots \\ X_{500,2}' \end{bmatrix}, X_3 = \begin{bmatrix} X_{1,3}' \\ \vdots \\ X_{500,3}' \end{bmatrix}, Y_1 = \begin{bmatrix} Y_{1,1} \\ \vdots \\ Y_{500,1} \end{bmatrix}, Y_2 = \begin{bmatrix} Y_{1,2} \\ \vdots \\ Y_{500,2} \end{bmatrix}$$

Our OLS estimates are therefore, using MATLAB,

$$\begin{aligned} \hat{\beta}_1^{OLS} &= (X_1'X_1)^{-1} X_1'Y_1 = \begin{bmatrix} 0.9897 \\ 1.0921 \\ 0.9505 \\ 0.9618 \end{bmatrix} \\ \hat{\beta}_2^{OLS} &= (X_2'X_2)^{-1} X_2'Y_2 = \begin{bmatrix} 0.9872 \\ 0.9262 \\ 0.9983 \\ 1.0360 \end{bmatrix} \\ \hat{\beta}_3^{OLS} &= (X_3'X_3)^{-1} X_3'Y_3 = \begin{bmatrix} 0.9836 \\ 1.0645 \\ 1.0269 \\ 0.8926 \\ 1.0550 \\ 1.0509 \end{bmatrix} \end{aligned}$$

(2) Provide a consistent estimate for Σ based on the regression estimates from (1).

Solution: Here, we have that

$$\Sigma = E\left[\varepsilon_i \varepsilon_j'\right] = \begin{bmatrix} \varepsilon_1\varepsilon_1' & \varepsilon_1\varepsilon_2' & \varepsilon_1\varepsilon_3' \\ \varepsilon_2\varepsilon_1' & \varepsilon_2\varepsilon_2' & \varepsilon_2\varepsilon_3' \\ \varepsilon_3\varepsilon_1' & \varepsilon_3\varepsilon_2' & \varepsilon_3\varepsilon_3' \end{bmatrix}$$

The consistent estimator for this matrix is

$$\hat{\Sigma} = \begin{bmatrix} \hat{\sigma}_{11}^{OLS} & \hat{\sigma}_{12}^{OLS} & \hat{\sigma}_{13}^{OLS} \\ \hat{\sigma}_{21}^{OLS} & \hat{\sigma}_{22}^{OLS} & \hat{\sigma}_{23}^{OLS} \\ \hat{\sigma}_{31}^{OLS} & \hat{\sigma}_{32}^{OLS} & \hat{\sigma}_{33}^{OLS} \end{bmatrix}$$

Where $\hat{\sigma}_{ij}^{OLS} = \frac{1}{n}\left(Y_i - X_i\hat{\beta}_i^{OLS}\right)'\left(Y_j - X_j\hat{\beta}_j^{OLS}\right)$. Using MATLAB, we get

$$\hat{\Sigma} = \begin{bmatrix} 3.7251 & 0.0822 & -0.1812 \\ 0.0822 & 3.5556 & -0.1226 \\ -0.1812 & -0.1226 & 4.1731 \end{bmatrix}$$

(3) Provide the standard error estimates for each of the three vectors of coefficient estimates obtained in (1).

Solution: Recall that the standard error estimates for the coefficients are given by

$$se\left(\hat{\beta}_{ji}^{OLS}\right) = \sqrt{\hat{\Sigma}_{jj}\left[\left(X_j'X_j\right)^{-1}\right]_{ii}}$$

Where $\hat{\beta}_{ji}^{OLS}$ is the OLS estimate for the ith coefficient of the jth equation.
Using MATLAB, we thus have

$$\begin{aligned} se\left(\hat{\beta}_{11}^{OLS}\right) &= 0.0464 \text{ ; } se\left(\hat{\beta}_{12}^{OLS}\right) = 0.0446 \text{ ; } se\left(\hat{\beta}_{13}^{OLS}\right) = 0.0465 \text{ ; } se\left(\hat{\beta}_{14}^{OLS}\right) = 0.0453 \\ se\left(\hat{\beta}_{21}^{OLS}\right) &= 0.0428 \text{ ; } se\left(\hat{\beta}_{22}^{OLS}\right) = 0.0439 \text{ ; } se\left(\hat{\beta}_{23}^{OLS}\right) = 0.0419 \text{ ; } se\left(\hat{\beta}_{24}^{OLS}\right) = 0.0427 \end{aligned}$$

$$\begin{aligned} se\left(\hat{\beta}_{31}^{OLS}\right) &= 0.0497 \text{ ; } se\left(\hat{\beta}_{32}^{OLS}\right) = 0.0477 \text{ ; } se\left(\hat{\beta}_{33}^{OLS}\right) = 0.0488 \text{ ; } \\ se\left(\hat{\beta}_{34}^{OLS}\right) &= 0.0494 \text{ ; } se\left(\hat{\beta}_{35}^{OLS}\right) = 0.0462 \text{ ; } se\left(\hat{\beta}_{36}^{OLS}\right) = 0.0480 \end{aligned}$$

(4) Provide the GLS estimates of the parameter vectors β_j, $j = 1, 2, 3$, using the estimate for Σ obtained in (2).

Solution: The variance-covariance matrix for the regression is

$$\Sigma \otimes I_{500}$$

If we write our model as

$$\begin{aligned} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} &= \begin{bmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_3 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \\ Y &= X\beta + \varepsilon \end{aligned}$$

Our FGLS estimates will be

$$\begin{bmatrix} \hat{\beta}_1^{FGLS} \\ \hat{\beta}_2^{FGLS} \\ \hat{\beta}_3^{FGLS} \end{bmatrix} = \left(X'\left(\Sigma \otimes I_{500}\right)^{-1} X\right)^{-1} X'\left(\Sigma \otimes I_{500}\right)^{-1} Y$$

Evaluating this expression in MATLAB gives us:

$$\hat{\beta}_1^{FGLS} = \begin{bmatrix} 0.9876 \\ 1.0921 \\ 0.9483 \\ 0.9674 \end{bmatrix}, \ \hat{\beta}_2^{FGLS} = \begin{bmatrix} 0.9859 \\ 0.9271 \\ 0.9970 \\ 1.0380 \end{bmatrix}, \ \hat{\beta}_3^{FGLS} = \begin{bmatrix} 0.9842 \\ 1.0629 \\ 1.0295 \\ 0.8895 \\ 1.0544 \\ 1.0529 \end{bmatrix}$$

(5) Provide the standard error estimates for each of the three vectors of coefficient estimates obtained in (4).

Solution: As in part (2), estimate

$$\hat{\Sigma}^{FGLS} = \begin{bmatrix} \hat{\sigma}_{11}^{FGLS} & \hat{\sigma}_{12}^{FGLS} & \hat{\sigma}_{13}^{FGLS} \\ \hat{\sigma}_{21}^{FGLS} & \hat{\sigma}_{22}^{FGLS} & \hat{\sigma}_{23}^{FGLS} \\ \hat{\sigma}_{31}^{FGLS} & \hat{\sigma}_{32}^{FGLS} & \hat{\sigma}_{33}^{FGLS} \end{bmatrix}$$

Where

$$\hat{\sigma}_{ij}^{FGLS} = \frac{1}{n}\left(Y_i - X_i\hat{\beta}_i^{FGLS}\right)'\left(Y_j - X_j\hat{\beta}_j^{FGLS}\right)$$

Using MATLAB, we have:

$$\hat{\Sigma}^{FGLS} = \begin{bmatrix} 3.7252 & 0.0842 & -0.1844 \\ 0.0842 & 3.5556 & -0.1235 \\ -0.1844 & -0.1235 & 4.1731 \end{bmatrix}$$

Defining the standard errors as in part (3), we have

$$se\left(\hat{\beta}_{ji}^{FGLS}\right) = \sqrt{\hat{\Sigma}_{jj}^{FGLS}\left[\left(X_j'X_j\right)^{-1}\right]_{ii}}$$

Where $\hat{\beta}_{ji}^{OLS}$ is the OLS estimate for the *ith* coefficient of the *jth* equation.
Using MATLAB, we thus have

$$\begin{aligned} se\left(\hat{\beta}_{11}^{FGLS}\right) &= 0.0464\ ; se\left(\hat{\beta}_{12}^{FGLS}\right) = 0.0446\ ; se\left(\hat{\beta}_{13}^{FGLS}\right) = 0.0465\ ; se\left(\hat{\beta}_{14}^{FGLS}\right) = 0.0453 \\ se\left(\hat{\beta}_{21}^{FGLS}\right) &= 0.0428\ ; se\left(\hat{\beta}_{22}^{FGLS}\right) = 0.0439\ ; se\left(\hat{\beta}_{23}^{FGLS}\right) = 0.0419\ ; se\left(\hat{\beta}_{24}^{FGLS}\right) = 0.0427 \end{aligned}$$

$$\begin{aligned} se\left(\hat{\beta}_{31}^{FGLS}\right) &= 0.0497\ ; se\left(\hat{\beta}_{32}^{FGLS}\right) = 0.0477\ ; se\left(\hat{\beta}_{33}^{FGLS}\right) = 0.0488\ ; \\ se\left(\hat{\beta}_{34}^{FGLS}\right) &= 0.0494\ ; se\left(\hat{\beta}_{35}^{FGLS}\right) = 0.0462\ ; se\left(\hat{\beta}_{36}^{FGLS}\right) = 0.0480 \end{aligned}$$

(6) Briefly discuss the results obtained in (1) and (4) for the point estimates for β_j, $j = 1, 2, 3$.

Solution: The results in (1) and (4) are very similar.

(7) Briefly discuss the results obtained in (3) and (5) for the standard error estimates for $\hat{\beta}_j$, $j = 1, 2, 3$.

Solution: At the number of decimal points MATLAB has rounded off to, we do not see any difference in the results for the standard error estimates. Presumably, since FGLS is more efficient than OLS asymptotically, we would expect that, with a large sample size (500 is probably sufficiently large), the standard errors for the GLS estimators would be smaller than those for the OLS estimators.

(8) Suppose now that it was claimed that the coefficients on x_1 and x_2 in the first and third regression should be the same, and that the coefficients on x_5 and x_6 in the second and third equations should be the same. Provide an estimate for all three parameter vectors that incorporate these constraints.

Solution: Here, insteading of writing our model as

$$\begin{aligned} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} &= \begin{bmatrix} x_1' & x_2' & x_3' & x_4' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_5' & x_6' & x_7' & x_8' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1' & x_2' & x_5' & x_6' & x_9' & x_{10}' \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \\ Y &= X\beta + \varepsilon \end{aligned}$$

we need only write our model as

$$\begin{aligned}
\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} &= \begin{bmatrix} x_1' & x_2' & x_3' & x_4' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_5' & x_6' & x_7' & x_8' & 0 & 0 \\ x_1' & x_2' & 0 & 0 & x_5' & x_6' & 0 & 0 & x_9' & x_{10}' \end{bmatrix} \beta + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \\
Y &= X^R \beta^R + \varepsilon^R
\end{aligned}$$

And compute $\beta = (X'X)^{-1} X'Y$. MATLAB gives the following result:

$$\hat{\beta}_1^R = \begin{bmatrix} 0.9870 \\ 1.0782 \\ 0.9576 \\ 0.9673 \end{bmatrix}, \ \hat{\beta}_2^R = \begin{bmatrix} 1.0052 \\ 0.9072 \\ 0.9978 \\ 1.0381 \end{bmatrix}, \ \hat{\beta}_3^R = \begin{bmatrix} 0.9870 \\ 1.0782 \\ 1.0052 \\ 0.9072 \\ 1.0510 \\ 1.0472 \end{bmatrix}$$

(9) Test the hypothesis of the claims made in (8).
Solution: To test the hypothesis

$$H_0 : \beta_{1,1} = \beta_{3,1}, \ \beta_{1,2} = \beta_{3,2}, \ \beta_{2,5} = \beta_{3,5}, \ \beta_{2,6} = \beta_{3,6}$$

I will use the following test statistic:

$$F = \frac{\hat{\varepsilon}_R'\hat{\varepsilon}_R - \hat{\varepsilon}_{UR}'\hat{\varepsilon}_{UR}}{\hat{\varepsilon}_{UR}'\hat{\varepsilon}_{UR}} \cdot \frac{n-k}{p}$$

Where

$$\begin{aligned}
\hat{\varepsilon}_R'\hat{\varepsilon}_R &= \left(Y - X^R \hat{\beta}^R\right)' \left(Y - X^R \hat{\beta}^R\right) \\
\hat{\varepsilon}_{UR}'\hat{\varepsilon}_{UR} &= (M_X Y)' (M_X Y) \\
n &= 500 \\
k &= 14 \\
p &= 4
\end{aligned}$$

Using MATLAB to compute this F statistic,

$$F = 0.0598$$

Since

$$c^*_{0.05, F(4,486)} = 2.39$$

We have that

$$F = 0.0598 \leq 2.39 = c^*_{0.05, F(4,486)}$$

And we therefore fail to reject the null hypothesis.

Question 3 (question 5, in Greene, page 423):
Consider the following model:

$$\begin{aligned} y_{1i} &= \gamma_1 y_{2i} + \beta_{11} x_{1i} + \varepsilon_{1i}, \\ y_{2i} &= \gamma_2 y_{1i} + \beta_{22} x_{2i} + \beta_{32} x_{3i} + \varepsilon_{2i}. \end{aligned}$$

All variables are measured as deviations from their means. The sample of 25 observations produces the following matrix of sums of squares and cross products:

$$\begin{array}{c|ccccc} & y_1 & y_2 & x_1 & x_2 & x_3 \\ \hline y_1 & 20 & 6 & 4 & 3 & 5 \\ y_2 & 6 & 10 & 3 & 6 & 7 \\ x_1 & 4 & 3 & 5 & 2 & 3 \\ x_2 & 3 & 6 & 2 & 10 & 8 \\ x_3 & 5 & 7 & 3 & 8 & 15 \end{array}$$

(1) Estimate the two equations by OLS.
Solution: Rewriting the two equations in a manner more amenable to OLS,

$$\begin{aligned} Y_1 &= \begin{bmatrix} Y_2 & X_1 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \beta_{11} \end{bmatrix} + \varepsilon_1 \\ Y_2 &= \begin{bmatrix} Y_1 & X_2 & X_3 \end{bmatrix} \begin{bmatrix} \gamma_2 \\ \beta_{22} \\ \beta_{32} \end{bmatrix} + \varepsilon_2 \end{aligned}$$

First, estimating the first equation, we have that

$$\begin{aligned} \begin{bmatrix} \hat{\gamma}_1 \\ \hat{\beta}_{11} \end{bmatrix} &= \left[\begin{bmatrix} Y_2' \\ X_1' \end{bmatrix} \begin{bmatrix} Y_2 & X_1 \end{bmatrix} \right]^{-1} \begin{bmatrix} Y_2' \\ X_1' \end{bmatrix} Y_1 \\ &= \begin{bmatrix} Y_2'Y_2 & Y_2'X_1 \\ X_1'Y_2 & X_1'X_1 \end{bmatrix}^{-1} \begin{bmatrix} Y_2'Y_1 \\ X_1'Y_1 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 3 \\ 3 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{18}{41} \\ \frac{22}{41} \end{bmatrix} \end{aligned}$$

Similarly, for the second equation,

$$\begin{aligned} \begin{bmatrix} \hat{\gamma}_2 \\ \hat{\beta}_{22} \\ \hat{\beta}_{32} \end{bmatrix} &= \left[\begin{bmatrix} Y_1' \\ X_2' \\ X_3' \end{bmatrix} \begin{bmatrix} Y_1 & X_2 & X_3 \end{bmatrix} \right]^{-1} \begin{bmatrix} Y_1' \\ X_2' \\ X_3' \end{bmatrix} Y_2 \\ &= \begin{bmatrix} Y_1'Y_1 & Y_1'X_2 & Y_1'X_3 \\ X_2'Y_1 & X_2'X_2 & X_2'X_3 \\ X_3'Y_1 & X_3'X_2 & X_3'X_3 \end{bmatrix}^{-1} \begin{bmatrix} Y_1'Y_2 \\ X_2'Y_2 \\ X_3'Y_2 \end{bmatrix} \\ &= \begin{bmatrix} 20 & 3 & 5 \\ 3 & 10 & 8 \\ 5 & 8 & 15 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} \frac{304}{1575} \\ \frac{121}{315} \\ \frac{311}{1575} \end{bmatrix} \end{aligned}$$

(2) Estimate the parameters of the two equations by 2SLS. Also, estimate the asymptotic covariance matrix of the 2SLS estimates.
Solution:
Equation 1: Step 1: Regress all the right-hand side variables $W = \begin{bmatrix} Y_2 & X_1 \end{bmatrix}$ on all the exogenous variables $Z = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$ and obtain estimates for the right-hand side variables $\hat{W}$.

$$\hat{W} = Z \left(Z'Z \right)^{-1} Z'W$$

Step 2: Estimate, using OLS, the parameters $\begin{bmatrix} \gamma_1 \\ \beta_{11} \end{bmatrix}$ of the model in which we replace the right-hand side variables with the estimates from step 1. That is, estimate $\begin{bmatrix} \gamma_1 \\ \beta_{11} \end{bmatrix}$ for

$$Y_1 = \hat{W} \begin{bmatrix} \gamma_1 \\ \beta_{11} \end{bmatrix} + \varepsilon$$

The estimator is

$$\begin{aligned}
\begin{bmatrix} \hat{\gamma}_{1,2SLS} \\ \hat{\beta}_{11,2SLS} \end{bmatrix} &= \left(\hat{W}'\hat{W}\right)^{-1} \hat{W}'Y_1 \\
&= \left(W'Z\left(Z'Z\right)^{-1} Z'Z\left(Z'Z\right)^{-1} Z'W\right)^{-1} W'Z\left(Z'Z\right)^{-1} Z'Y_1 \\
&= \left(\left(Z'W\right)'\left(Z'Z\right)^{-1} Z'W\right)^{-1} \left(Z'W\right)'\left(Z'Z\right)^{-1} Z'Y_1 \\
&= \left(\left(\begin{bmatrix} X_1' \\ X_2' \\ X_3' \end{bmatrix} \begin{bmatrix} Y_2 & X_1 \end{bmatrix}\right)' \left(\begin{bmatrix} X_1' \\ X_2' \\ X_3' \end{bmatrix} \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}\right)^{-1} \begin{bmatrix} X_1' \\ X_2' \\ X_3 \end{bmatrix} \begin{bmatrix} Y_2 & X_1 \end{bmatrix}\right) \\
&\quad \cdot \left(\begin{bmatrix} X_1' \\ X_2' \\ X_3' \end{bmatrix} \begin{bmatrix} Y_2 & X_1 \end{bmatrix}\right)' \left(\begin{bmatrix} X_1' \\ X_2' \\ X_3' \end{bmatrix} \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}\right)^{-1} \begin{bmatrix} X_1'Y_1 \\ X_2'Y_1 \\ X_3Y_1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \left(\begin{bmatrix} X_1'Y_2 & X_1'X_1 \\ X_2'Y_2 & X_2'X_1 \\ X_3'Y_2 & X_3'X_1 \end{bmatrix}' \begin{bmatrix} X_1'X_1 & X_1'X_2 & X_1'X_3 \\ X_2'X_1 & X_2'X_2 & X_2'X_3 \\ X_3'X_1 & X_3'X_2 & X_3'X_3 \end{bmatrix}^{-1} \begin{bmatrix} X_1'Y_2 & X_1'X_1 \\ X_2'Y_2 & X_2'X_1 \\ X_3'Y_2 & X_3'X_1 \end{bmatrix}\right)^{-1} \\
&\quad \cdot \begin{bmatrix} X_1'Y_2 & X_1'X_1 \\ X_2'Y_2 & X_2'X_1 \\ X_3'Y_2 & X_3'X_1 \end{bmatrix}' \begin{bmatrix} X_1'X_1 & X_1'X_2 & X_1'X_3 \\ X_2'X_1 & X_2'X_2 & X_2'X_3 \\ X_3'X_1 & X_3'X_2 & X_3'X_3 \end{bmatrix}^{-1} \begin{bmatrix} X_1'Y_1 \\ X_2'Y_1 \\ X_3Y_1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \left(\begin{bmatrix} 3 & 6 & 7 \\ 5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 5 \\ 6 & 2 \\ 7 & 3 \end{bmatrix}\right)^{-1} \begin{bmatrix} 3 & 6 & 7 \\ 5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} \\
&= \begin{bmatrix} \frac{246}{667} \\ \frac{386}{667} \end{bmatrix}
\end{aligned}$$

Finally, for the estimated variance-covariance matrix of $\begin{bmatrix} \hat{\gamma}_{1,2SLS} \\ \hat{\beta}_{11,2SLS} \end{bmatrix}$, note that

$$\begin{aligned}
Var\left(\begin{bmatrix} \hat{\gamma}_{1,2SLS} \\ \hat{\beta}_{11,2SLS} \end{bmatrix} \middle| W, Z\right) &= Var\left(\left(\hat{W}'\hat{W}\right)^{-1} \hat{W}'Y_1 \middle| W, Z\right) \\
&= \left(\hat{W}'\hat{W}\right)^{-1} \hat{W} Var\left(Y_1 | W, Z\right) \hat{W}' \left(\hat{W}'\hat{W}\right)^{-1}
\end{aligned}$$

If we assume conditional homoskedasticity, we have

$$\begin{aligned}
Var\left(\begin{bmatrix} \hat{\gamma}_{1,2SLS} \\ \hat{\beta}_{11,2SLS} \end{bmatrix} \middle| W, Z\right) &= \sigma^2 \left(\hat{W}'\hat{W}\right)^{-1} \\
Var\left(\begin{bmatrix} \hat{\gamma}_{1,2SLS} \\ \hat{\beta}_{11,2SLS} \end{bmatrix}\right) &= \sigma^2 \left(\hat{W}'\hat{W}\right)^{-1}
\end{aligned}$$

Since this is not feasible, we estimate it using

$$\widehat{Var\left(\begin{bmatrix}\hat{\gamma}_{1,2SLS}\\ \hat{\beta}_{11,2SLS}\end{bmatrix}\right)} = \hat{\sigma}^2_{2SLS}\left(\hat{W}'\hat{W}\right)^{-1}$$

Where

$$\begin{aligned}
\hat{\sigma}^2_{2SLS} &= \frac{1}{n}\left(Y_1 - W\begin{bmatrix}\hat{\gamma}_{1,2SLS}\\ \hat{\beta}_{11,2SLS}\end{bmatrix}\right)'\left(Y_1 - W\begin{bmatrix}\hat{\gamma}_{1,2SLS}\\ \hat{\beta}_{11,2SLS}\end{bmatrix}\right)\\
&= \frac{1}{n}\left(Y_1'Y_1 - 2Y_1'W\begin{bmatrix}\hat{\gamma}_{1,2SLS}\\ \hat{\beta}_{11,2SLS}\end{bmatrix} + \begin{bmatrix}\hat{\gamma}_{1,2SLS} & \hat{\beta}_{11,2SLS}\end{bmatrix}W'W\begin{bmatrix}\hat{\gamma}_{1,2SLS}\\ \hat{\beta}_{11,2SLS}\end{bmatrix}\right)\\
&= \frac{1}{25}\left(\begin{array}{c} Y_1'Y_1 - 2\begin{bmatrix}Y_1'Y_2 & Y_1'X_1\end{bmatrix}\begin{bmatrix}\hat{\gamma}_{1,2SLS}\\ \hat{\beta}_{11,2SLS}\end{bmatrix}\\ +\begin{bmatrix}\hat{\gamma}_{1,2SLS} & \hat{\beta}_{11,2SLS}\end{bmatrix}\begin{bmatrix}Y_2'Y_2 & Y_2'X_1\\ X_1'Y_2 & X_1'X_1\end{bmatrix}\begin{bmatrix}\hat{\gamma}_{1,2SLS}\\ \hat{\beta}_{11,2SLS}\end{bmatrix}\end{array}\right)\\
&= \frac{1}{25}\left(20 - 2\begin{bmatrix}6 & 4\end{bmatrix}\begin{bmatrix}\frac{246}{667}\\ \frac{386}{667}\end{bmatrix} + \begin{bmatrix}\frac{246}{667} & \frac{386}{667}\end{bmatrix}\begin{bmatrix}10 & 3\\ 3 & 5\end{bmatrix}\begin{bmatrix}\frac{246}{667}\\ \frac{386}{667}\end{bmatrix}\right)\\
&= \frac{6788976.}{11122225} \approx 0.6104
\end{aligned}$$

Also,

$$\begin{aligned}
\left(\hat{W}'\hat{W}\right)^{-1} &= \left(\begin{bmatrix}3 & 6 & 7\\ 5 & 2 & 3\end{bmatrix}\begin{bmatrix}5 & 2 & 3\\ 2 & 10 & 8\\ 3 & 8 & 15\end{bmatrix}^{-1}\begin{bmatrix}3 & 5\\ 6 & 2\\ 7 & 3\end{bmatrix}\right)^{-1}\\
&= \begin{bmatrix}\frac{235}{667} & -\frac{141}{667}\\ -\frac{141}{667} & \frac{218}{667}\end{bmatrix}
\end{aligned}$$

Therefore, the estimated variance-covariance matrix is

$$\begin{aligned}
\hat{V} &= \hat{\sigma}^2_{2SLS}\left(\hat{W}'\hat{W}\right)^{-1}\\
&= 0.6104\begin{bmatrix}\frac{235}{667} & -\frac{141}{667}\\ -\frac{141}{667} & \frac{218}{667}\end{bmatrix}\\
&= \begin{bmatrix}0.2151 & -0.1290\\ -0.1290 & 0.1995\end{bmatrix}
\end{aligned}$$

Equation 2: Step 1: Regress all the right-hand side variables $W = \begin{bmatrix}Y_1 & X_2 & X_3\end{bmatrix}$ on all the exogenous variables $Z = \begin{bmatrix}X_1 & X_2 & X_3\end{bmatrix}$ and obtain estimates for the right-hand side variables $\hat{W}$.

$$\hat{W} = Z\left(Z'Z\right)^{-1}Z'W$$

Step 2: Estimate, using OLS, the parameters $\begin{bmatrix}\gamma_2\\ \beta_{22}\\ \beta_{32}\end{bmatrix}$ of the model in which we replace the right-hand side variables with the estimates from step 1. That is, estimate $\begin{bmatrix}\gamma_2\\ \beta_{22}\\ \beta_{32}\end{bmatrix}$ for

$$Y_2 = \hat{W}\begin{bmatrix}\gamma_2\\ \beta_{22}\\ \beta_{32}\end{bmatrix} + \varepsilon$$

The estimator is

$$\begin{aligned}\begin{bmatrix}\hat{\gamma}_{2,2SLS}\\ \hat{\beta}_{22,2SLS}\\ \hat{\beta}_{32,2SLS}\end{bmatrix} &= \left(\hat{W}'\hat{W}\right)^{-1}\hat{W}'Y_2\\ &= \left((Z'W)'\left(Z'Z\right)^{-1}Z'Z\left(Z'Z\right)^{-1}Z'W\right)^{-1}(Z'W)'\left(Z'Z\right)^{-1}Z'Y_2\\ &= \left((Z'W)'\left(Z'Z\right)^{-1}Z'W\right)^{-1}(Z'W)'\left(Z'Z\right)^{-1}Z'Y_2\end{aligned}$$

Here, we have that

$$\begin{aligned}Z'W &= \begin{bmatrix}X_1'\\ X_2'\\ X_3'\end{bmatrix}\begin{bmatrix}Y_1 & X_2 & X_3\end{bmatrix} = \begin{bmatrix}X_1'Y_1 & X_1'X_2 & X_1'X_3\\ X_2'Y_1 & X_2'X_2 & X_2'X_3\\ X_3'Y_1 & X_3'X_2 & X_3'X_3\end{bmatrix}\\ &= \begin{bmatrix}4 & 2 & 3\\ 3 & 10 & 8\\ 5 & 8 & 15\end{bmatrix}\end{aligned}$$

$$\begin{aligned}Z'Z &= \begin{bmatrix}X_1'\\ X_2'\\ X_3'\end{bmatrix}\begin{bmatrix}X_1 & X_2 & X_3\end{bmatrix} = \begin{bmatrix}X_1'X_1 & X_1'X_2 & X_1'X_3\\ X_2'X_1 & X_2'X_2 & X_2'X_3\\ X_3'X_1 & X_3'X_2 & X_3'X_3\end{bmatrix}\\ &= \begin{bmatrix}5 & 2 & 3\\ 2 & 10 & 8\\ 3 & 8 & 15\end{bmatrix}\end{aligned}$$

$$Z'Y_2 = \begin{bmatrix}X_1'Y_2\\ X_2'Y_2\\ X_3'Y_2\end{bmatrix} = \begin{bmatrix}3\\ 6\\ 7\end{bmatrix}$$

Putting this all together,

$$\begin{aligned}\begin{bmatrix}\hat{\gamma}_{2,2SLS}\\ \hat{\beta}_{22,2SLS}\\ \hat{\beta}_{32,2SLS}\end{bmatrix} &= \left((Z'W)'\left(Z'Z\right)^{-1}Z'W\right)^{-1}(Z'W)'\left(Z'Z\right)^{-1}Z'Y_2\\ &= \left(\begin{bmatrix}4 & 3 & 5\\ 2 & 10 & 8\\ 3 & 8 & 15\end{bmatrix}\begin{bmatrix}5 & 2 & 3\\ 2 & 10 & 8\\ 3 & 8 & 15\end{bmatrix}^{-1}\begin{bmatrix}4 & 2 & 3\\ 3 & 10 & 8\\ 5 & 8 & 15\end{bmatrix}\right)^{-1}\\ &\quad\cdot\begin{bmatrix}4 & 3 & 5\\ 2 & 10 & 8\\ 3 & 8 & 15\end{bmatrix}\begin{bmatrix}5 & 2 & 3\\ 2 & 10 & 8\\ 3 & 8 & 15\end{bmatrix}^{-1}\begin{bmatrix}3\\ 6\\ 7\end{bmatrix}\\ &= \begin{bmatrix}\frac{31}{64}\\ \frac{47}{128}\\ \frac{7}{64}\end{bmatrix}\end{aligned}$$

Finally, for the estimated variance-covariance matrix of $\begin{bmatrix}\hat{\gamma}_{2,2SLS}\\ \hat{\beta}_{22,2SLS}\\ \hat{\beta}_{32,2SLS}\end{bmatrix}$, note that

$$\begin{aligned}Var\left(\left.\begin{bmatrix}\hat{\gamma}_{2,2SLS}\\ \hat{\beta}_{22,2SLS}\\ \hat{\beta}_{32,2SLS}\end{bmatrix}\right| W,Z\right) &= Var\left(\left.\left(\hat{W}'\hat{W}\right)^{-1}\hat{W}'Y_2\right| W,Z\right)\\ &= \left(\hat{W}'\hat{W}\right)^{-1}\hat{W}Var\left(Y_1|\,W,Z\right)\hat{W}'\left(\hat{W}'\hat{W}\right)^{-1}\end{aligned}$$

If we assume conditional homoskedasticity, we have

$$\begin{aligned}
Var\left(\left.\begin{bmatrix}\hat{\gamma}_{2,2SLS}\\ \hat{\beta}_{22,2SLS}\\ \hat{\beta}_{32,2SLS}\end{bmatrix}\right| W, Z\right) &= \sigma^2\left(\hat{W}'\hat{W}\right)^{-1}\\
Var\left(\begin{bmatrix}\hat{\gamma}_{2,2SLS}\\ \hat{\beta}_{22,2SLS}\\ \hat{\beta}_{32,2SLS}\end{bmatrix}\right) &= \sigma^2\left(\hat{W}'\hat{W}\right)^{-1}
\end{aligned}$$

Since this is not feasible, we estimate it using

$$Var\left(\widehat{\begin{bmatrix}\hat{\gamma}_{2,2SLS}\\ \hat{\beta}_{22,2SLS}\\ \hat{\beta}_{32,2SLS}\end{bmatrix}}\right) = \hat{\sigma}^2_{2SLS}\left(\hat{W}'\hat{W}\right)^{-1}$$

Where

$$\begin{aligned}
\hat{\sigma}^2_{2SLS} &= \frac{1}{n}\left(Y_2 - W\begin{bmatrix}\hat{\gamma}_{2,2SLS}\\ \hat{\beta}_{22,2SLS}\\ \hat{\beta}_{32,2SLS}\end{bmatrix}\right)'\left(Y_2 - W\begin{bmatrix}\hat{\gamma}_{2,2SLS}\\ \hat{\beta}_{22,2SLS}\\ \hat{\beta}_{32,2SLS}\end{bmatrix}\right)\\
&= \frac{1}{n}\left(Y_2'Y_2 - 2Y_2'W\begin{bmatrix}\hat{\gamma}_{2,2SLS}\\ \hat{\beta}_{22,2SLS}\\ \hat{\beta}_{32,2SLS}\end{bmatrix} + \begin{bmatrix}\hat{\gamma}_{2,2SLS} & \hat{\beta}_{22,2SLS} & \hat{\beta}_{32,2SLS}\end{bmatrix}W'W\begin{bmatrix}\hat{\gamma}_{2,2SLS}\\ \hat{\beta}_{22,2SLS}\\ \hat{\beta}_{32,2SLS}\end{bmatrix}\right)\\
&= \frac{1}{25}\left(\begin{array}{c} Y_2'Y_2 - 2\begin{bmatrix}Y_2'Y_1 & Y_2'X_2 & Y_2'X_3\end{bmatrix}\begin{bmatrix}\hat{\gamma}_{2,2SLS}\\ \hat{\beta}_{22,2SLS}\\ \hat{\beta}_{32,2SLS}\end{bmatrix}\\ +\begin{bmatrix}\hat{\gamma}_{2,2SLS} & \hat{\beta}_{22,2SLS} & \hat{\beta}_{32,2SLS}\end{bmatrix}\begin{bmatrix}Y_1'Y_1 & Y_1'X_2 & Y_1'X_3\\ X_2'Y_1 & X_2'X_2 & X_2'X_3\\ X_3'Y_1 & X_3'X_2 & X_3'X_3\end{bmatrix}\begin{bmatrix}\hat{\gamma}_{2,2SLS}\\ \hat{\beta}_{22,2SLS}\\ \hat{\beta}_{32,2SLS}\end{bmatrix}\end{array}\right)\\
&= \frac{1}{25}\left(10 - 2\begin{bmatrix}6 & 6 & 7\end{bmatrix}\begin{bmatrix}\frac{31}{64}\\ \frac{47}{128}\\ \frac{7}{64}\end{bmatrix} + \begin{bmatrix}\frac{31}{64} & \frac{47}{128} & \frac{7}{64}\end{bmatrix}\begin{bmatrix}20 & 3 & 5\\ 3 & 10 & 8\\ 5 & 8 & 15\end{bmatrix}\begin{bmatrix}\frac{31}{64}\\ \frac{47}{128}\\ \frac{7}{64}\end{bmatrix}\right)\\
&= \frac{10993}{40960} \approx 0.2684
\end{aligned}$$

Also,

$$\begin{aligned}
\left(\hat{W}'\hat{W}\right)^{-1} &= \left(\begin{bmatrix}4 & 3 & 5\\ 2 & 10 & 8\\ 3 & 8 & 15\end{bmatrix}\begin{bmatrix}5 & 2 & 3\\ 2 & 10 & 8\\ 3 & 8 & 15\end{bmatrix}^{-1}\begin{bmatrix}4 & 2 & 3\\ 3 & 10 & 8\\ 5 & 8 & 15\end{bmatrix}\right)^{-1}\\
&= \begin{bmatrix}\frac{2021}{4096} & -\frac{235}{8192} & -\frac{611}{4096}\\ -\frac{235}{8192} & \frac{2885}{16\,384} & -\frac{691}{8192}\\ -\frac{611}{4096} & -\frac{691}{8192} & \frac{661}{4096}\end{bmatrix}
\end{aligned}$$

Therefore, the estimated variance-covariance matrix is

$$\begin{aligned}
\hat{V} &= \hat{\sigma}^2_{2SLS}\left(\hat{W}'\hat{W}\right)^{-1}\\
&= 0.2684\begin{bmatrix}\frac{2021.}{4096} & -\frac{235}{8192} & -\frac{611}{4096}\\ -\frac{235}{8192} & \frac{2885}{16\,384} & -\frac{691}{8192}\\ -\frac{611}{4096} & -\frac{691}{8192} & \frac{661}{4096}\end{bmatrix}\\
&= \begin{bmatrix}0.1324 & -0.0077 & -0.0400\\ -0.0077 & 0.0473 & -0.0226\\ -0.0400 & -0.0226 & 0.0433\end{bmatrix}
\end{aligned}$$

(3) Obtain the LIML estimates of the parameters of the first equation.
Solution: The LIML estimates for the first equation can be computed as follows:

$$W_1^0 = E_1^{0\prime} E_1^0$$

Where we define

$$\begin{aligned} Y_1^0 &= \begin{bmatrix} y_1 & y_2 \end{bmatrix} \\ E_1^0 &= M_{x_1} Y_1^0 = \left[I - x_1 \left(x_1' x_1 \right)^{-1} x_1' \right] Y_1^0 \\ &= \begin{bmatrix} y_1 - x_1 \left(x_1' x_1 \right)^{-1} x_1' y_1 & y_2 - x_1 \left(x_1' x_1 \right)^{-1} x_1' y_2 \end{bmatrix} \\ &= \begin{bmatrix} y_1 - P_{x_1} y_1 & y_2 - P_{x_1} y_2 \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon_1^0 & \varepsilon_2^0 \end{bmatrix} \end{aligned}$$

This gives us:

$$W_1^0 = \begin{bmatrix} \varepsilon_1^{0\prime} \varepsilon_1^0 & \varepsilon_1^{0\prime} \varepsilon_2^0 \\ \varepsilon_2^{0\prime} \varepsilon_1^0 & \varepsilon_2^{0\prime} \varepsilon_2^0 \end{bmatrix}$$

Additonally, define

$$W_1^1 = E_1^{1\prime} E_1^1 = Y_1^{0\prime} \left[I - X \left(X' X \right)^{-1} X' \right] Y_1^0$$

Where

$$\begin{aligned} E_1^1 &= M_X Y_1^0 = \left[I - X \left(X' X \right)^{-1} X' \right] Y_1^0 \\ &= \begin{bmatrix} y_1 - X \left(X' X \right)^{-1} X' & y_2 - X \left(X' X \right)^{-1} X' y_2 \end{bmatrix} \\ &= \begin{bmatrix} y_1 - P_X y_1 & y_2 - P_X y_2 \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon_1^1 & \varepsilon_2^1 \end{bmatrix} \end{aligned}$$

And

$$X = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

Therefore, we obtain

$$W_1^1 = \begin{bmatrix} \varepsilon_1^{1\prime} \varepsilon_1^1 & \varepsilon_1^{1\prime} \varepsilon_2^1 \\ \varepsilon_2^{1\prime} \varepsilon_1^1 & \varepsilon_2^{1\prime} \varepsilon_2^1 \end{bmatrix}$$

Additionally, we need to define following eigenvalues

$$\lambda_1 = \text{smallest characteristic root of } \left(W_1^1 \right)^{-1} W_1^0$$

The estimator is then

$$\begin{aligned} \hat{\gamma}_{1,LIML} &= \left[\left(\varepsilon_2^{0\prime} \varepsilon_2^0 \right) - \lambda_1 \left(\varepsilon_2^{1\prime} \varepsilon_2^1 \right) \right]^{-1} \left[\varepsilon_1^{0\prime} \varepsilon_2^0 - \lambda_1 \left(\varepsilon_1^{1\prime} \varepsilon_2^1 \right) \right] \\ \hat{\beta}_{1,LIML} &= \left(x_1' x_1 \right)^{-1} x_1' \left(y_1 - y_2 \hat{\gamma}_{1,LIML} \right) \end{aligned}$$

Using MATLAB, we get

$$\begin{aligned} \hat{\gamma}_{1,LIML} &= 0.3671 \\ \hat{\beta}_{1,LIML} &= 0.5797 \end{aligned}$$

(4) Estimate the two equations by 3SLS.

Solution: In order to estimates the two equations by 3SLS we first need to estimate the variance-covariance matrix using the coefficient estimates we acquired in part (2):

$$\hat{\Sigma} = \begin{bmatrix} \frac{1}{n}\hat{\varepsilon}_1'\hat{\varepsilon}_1 & \frac{1}{n}\hat{\varepsilon}_1'\varepsilon_2 \\ \frac{1}{n}\hat{\varepsilon}_2'\hat{\varepsilon}_1 & \frac{1}{n}\hat{\varepsilon}_2'\hat{\varepsilon}_2 \end{bmatrix}$$

From part (2), we have that

$$\begin{aligned} \frac{1}{n}\hat{\varepsilon}_1'\hat{\varepsilon}_1 &= 0.6104 \\ \frac{1}{n}\hat{\varepsilon}_2'\hat{\varepsilon}_2 &= 0.2684 \end{aligned}$$

We need only calculate $\frac{1}{n}\hat{\varepsilon}_1'\hat{\varepsilon}_2$

$$\begin{aligned}
\frac{1}{n}\hat{\varepsilon}_1'\hat{\varepsilon}_2 &= \frac{1}{n}\left(Y_1 - W_1\begin{bmatrix} \hat{\gamma}_{1,2SLS} \\ \hat{\beta}_{11,2SLS} \end{bmatrix}\right)'\left(Y_2 - W_2\begin{bmatrix} \hat{\gamma}_{2,2SLS} \\ \hat{\beta}_{22,2SLS} \\ \hat{\beta}_{32,2SLS} \end{bmatrix}\right) \\
&= \frac{1}{n}\left(\begin{array}{c} Y_1'Y_2 - \begin{bmatrix} \hat{\gamma}_{1,2SLS} & \hat{\beta}_{11,2SLS} \end{bmatrix} W_1'Y_2 - Y_1'W_2\begin{bmatrix} \hat{\gamma}_{2,2SLS} \\ \hat{\beta}_{22,2SLS} \\ \hat{\beta}_{32,2SLS} \end{bmatrix} \\ + \begin{bmatrix} \hat{\gamma}_{1,2SLS} & \hat{\beta}_{11,2SLS} \end{bmatrix} W_1'W_2\begin{bmatrix} \hat{\gamma}_{2,2SLS} \\ \hat{\beta}_{22,2SLS} \\ \hat{\beta}_{32,2SLS} \end{bmatrix} \end{array}\right) \\
&= \frac{1}{n}\left(\begin{array}{c} Y_1'Y_2 - \begin{bmatrix} \hat{\gamma}_{1,2SLS} & \hat{\beta}_{11,2SLS} \end{bmatrix}\begin{bmatrix} Y_2'Y_2 \\ X_1'Y_2 \end{bmatrix} - \begin{bmatrix} Y_1'Y_1 & Y_1'X_2 & Y_1'X_3 \end{bmatrix}\begin{bmatrix} \hat{\gamma}_{2,2SLS} \\ \hat{\beta}_{22,2SLS} \\ \hat{\beta}_{32,2SLS} \end{bmatrix} \\ + \begin{bmatrix} \hat{\gamma}_{1,2SLS} & \hat{\beta}_{11,2SLS} \end{bmatrix}\begin{bmatrix} Y_2'Y_1 & Y_2'X_2 & Y_2'X_3 \\ X_1'Y_1 & X_1'X_2 & X_1'X_3 \end{bmatrix}\begin{bmatrix} \hat{\gamma}_{2,2SLS} \\ \hat{\beta}_{22,2SLS} \\ \hat{\beta}_{32,2SLS} \end{bmatrix} \end{array}\right) \\
&= \frac{1}{25}\left(\begin{array}{c} 6 - \begin{bmatrix} \frac{246}{667} & \frac{386}{667} \end{bmatrix}\begin{bmatrix} 10 \\ 3 \end{bmatrix} - \begin{bmatrix} 20 & 3 & 5 \end{bmatrix}\begin{bmatrix} \frac{31}{64} \\ \frac{47}{128} \\ \frac{7}{64} \end{bmatrix} \\ + \begin{bmatrix} \frac{246}{667} & \frac{386}{667} \end{bmatrix}\begin{bmatrix} 6 & 6 & 7 \\ 4 & 2 & 3 \end{bmatrix}\begin{bmatrix} \frac{31}{64} \\ \frac{47}{128} \\ \frac{7}{64} \end{bmatrix} \end{array}\right) \\
&= -\frac{585449}{2134400} \approx -0.2743
\end{aligned}$$

This gives us

$$\Sigma = \begin{bmatrix} 0.6104 & -0.2743 \\ -0.2743 & 0.2684 \end{bmatrix}$$

Now lets define the following variables:

$$\begin{aligned}
X &= \begin{bmatrix} y_2 & x_1 & 0 & 0 & 0 \\ 0 & 0 & y_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}, Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \\
Z &= \begin{bmatrix} x_1 & x_2 & x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}, \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \\
\hat{V} &= \hat{\Sigma} \otimes I_n \\
\hat{X}' &\equiv X'Z\left(Z'Z\right)^{-1}Z' = X'P_z
\end{aligned}$$

Then we can express the 3SLS estimator for α by:$\alpha_{3SLS} = \left(\hat{X}'\hat{V}^{-1}X\right)^{-1}\hat{X}'\hat{V}^{-1}Y$
After defining each element, we can find the estimator:

$$\begin{aligned}
V^{-1} &= \hat{\Sigma}^{-1}\otimes I_n \\
\hat{\Sigma}^{-1} &= \begin{bmatrix} \hat{\sigma}^{11} & \hat{\sigma}^{12} \\ \hat{\sigma}^{21} & \hat{\sigma}^{22} \end{bmatrix} \\
\hat{X}'\hat{V}^{-1}X &= \begin{bmatrix} \hat{\sigma}^{11}\hat{X}_1'X_1 & \hat{\sigma}^{12}\hat{X}_1'X_2 \\ \hat{\sigma}^{21}\hat{X}_2'X_1 & \hat{\sigma}^{22}\hat{X}_2'X_2 \end{bmatrix} \\
&= \begin{bmatrix} \hat{\sigma}^{11}X_1'Z_1\left(Z_1'Z_1\right)^{-1}Z_1'X_1 & \hat{\sigma}^{12}X_1'Z_1\left(Z_1'Z_1\right)^{-1}Z_1'X_2 \\ \hat{\sigma}^{21}X_2'Z_1\left(Z_1'Z_1\right)^{-1}Z_1'X_1 & \hat{\sigma}^{22}X_2'Z_1\left(Z_1'Z_1\right)^{-1}Z_1'X_2 \end{bmatrix} \\
\hat{X}'\hat{V}^{-1}Y &= \begin{bmatrix} \hat{\sigma}^{11}\hat{X}_1'y_1 \\ \hat{\sigma}^{22}\hat{X}_2'y_2 \end{bmatrix} = \begin{bmatrix} \hat{\sigma}^{11}X_1'Z_1\left(Z_1'Z_1\right)^{-1}Z_1'y_1 \\ \hat{\sigma}^{22}X_2'Z_1\left(Z_1'Z_1\right)^{-1}Z_1'y_2 \end{bmatrix}
\end{aligned}$$

Finally we obtain the 3SLS estimate, for which I used MATLAB:

$$\alpha_{3SLS} = \begin{bmatrix} \hat{\gamma}_{1,3SLS} \\ \hat{\beta}_{11,3SLS} \\ \hat{\gamma}_{2,3SLS} \\ \hat{\beta}_{22,3SLS} \\ \hat{\beta}_{32,3SLS} \end{bmatrix} = \begin{bmatrix} 0.36882 \\ 0.57871 \\ 0.47151 \\ 0.3104 \\ 0.16439 \end{bmatrix}$$

(5) Estimate the reduced-form coefficient matrix by OLS, and indirectly by using the structural estimates from part (2).

Solution: Solving for the reduced-form equations,

$$\begin{aligned}
Y_1 &= Y_2\gamma_1 + X_1\beta_{11} + \varepsilon_1 \\
Y_2 &= Y_1\gamma_2 + X_2\beta_{22} + X_3\beta_{32} + \varepsilon_2
\end{aligned}$$

$$\begin{aligned}
Y_1 &= \left(Y_1\gamma_2 + X_2\beta_{22} + X_3\beta_{32} + \varepsilon_2\right)\gamma_1 + X_1\beta_{11} + \varepsilon_1 \\
&= Y_1\gamma_2\gamma_1 + X_2\beta_{22}\gamma_1 + X_3\beta_{32}\gamma_1 + X_1\beta_{11} + \varepsilon_1 + \varepsilon_2\gamma_1
\end{aligned}$$

$$\begin{aligned}
Y_1\left(1-\gamma_1\gamma_2\right) &= X_2\beta_{22}\gamma_1 + X_3\beta_{32}\gamma_1 + X_1\beta_{11} + \varepsilon_1 + \varepsilon_2\gamma_1 \\
Y_1 &= X_1\frac{\beta_{11}}{1-\gamma_1\gamma_2} + X_2\frac{\beta_{22}\gamma_1}{1-\gamma_1\gamma_2} + X_3\frac{\beta_{32}\gamma_1}{1-\gamma_1\gamma_2} + \frac{\varepsilon_1+\varepsilon_2\gamma_1}{1-\gamma_1\gamma_2}
\end{aligned}$$

And

$$\begin{aligned}
Y_2 &= \left(X_2\frac{\beta_{22}\gamma_1}{1-\gamma_1\gamma_2} + X_3\frac{\beta_{32}\gamma_1}{1-\gamma_1\gamma_2} + X_1\frac{\beta_{11}}{1-\gamma_1\gamma_2} + \frac{\varepsilon_1+\varepsilon_2\gamma_1}{1-\gamma_1\gamma_2}\right)\gamma_2 + X_2\beta_{22} + X_3\beta_{32} + \varepsilon_2 \\
&= X_2\frac{\beta_{22}\gamma_1\gamma_2}{1-\gamma_1\gamma_2} + X_3\frac{\beta_{32}\gamma_1\gamma_2}{1-\gamma_1\gamma_2} + X_1\frac{\beta_{11}\gamma_2}{1-\gamma_1\gamma_2} + \frac{\varepsilon_1\gamma_2+\varepsilon_2\gamma_1\gamma_2}{1-\gamma_1\gamma_2} + X_2\beta_{22} + X_3\beta_{32} + \varepsilon_2 \\
&= X_1\frac{\beta_{11}\gamma_2}{1-\gamma_1\gamma_2} + X_2\left(\frac{\beta_{22}\gamma_1\gamma_2}{1-\gamma_1\gamma_2} + \beta_{22}\right) + X_3\left(\frac{\beta_{32}\gamma_1\gamma_2}{1-\gamma_1\gamma_2} + \beta_{32}\right) + \frac{\varepsilon_1\gamma_2+\varepsilon_2\gamma_1\gamma_2}{1-\gamma_1\gamma_2} + \varepsilon_2 \\
&= X_1\frac{\beta_{11}\gamma_2}{1-\gamma_1\gamma_2} + X_2\frac{\beta_{22}}{1-\gamma_1\gamma_2} + X_3\frac{\beta_{32}}{1-\gamma_1\gamma_2} + \frac{\varepsilon_1\gamma_2+\varepsilon_2}{1-\gamma_1\gamma_2}
\end{aligned}$$

That is,

$$\begin{aligned}
Y_1 &= X_1\frac{\beta_{11}}{1-\gamma_1\gamma_2} + X_2\frac{\beta_{22}\gamma_1}{1-\gamma_1\gamma_2} + X_3\frac{\beta_{32}\gamma_1}{1-\gamma_1\gamma_2} + \frac{\varepsilon_1+\varepsilon_2\gamma_1}{1-\gamma_1\gamma_2} \\
Y_2 &= X_1\frac{\beta_{11}\gamma_2}{1-\gamma_1\gamma_2} + X_2\frac{\beta_{22}}{1-\gamma_1\gamma_2} + X_3\frac{\beta_{32}}{1-\gamma_1\gamma_2} + \frac{\varepsilon_1\gamma_2+\varepsilon_2}{1-\gamma_1\gamma_2}
\end{aligned}$$

Or

$$
\begin{aligned}
Y_1 &= X_1\pi_{11} + X_2\pi_{12} + X_3\pi_{13} + u_1 \\
Y_2 &= X_1\pi_{21} + X_2\pi_{22} + X_3\pi_{23} + u_2
\end{aligned}
$$

Estimating equation 1 using OLS,

$$
\begin{aligned}
\begin{bmatrix} \hat{\pi}_{11} \\ \hat{\pi}_{12} \\ \hat{\pi}_{13} \end{bmatrix} &= \left(\begin{bmatrix} X_1' \\ X_2' \\ X_3' \end{bmatrix} \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} \right)^{-1} \begin{bmatrix} X_1' \\ X_2' \\ X_3' \end{bmatrix} Y_1 \\
&= \begin{bmatrix} X_1'X_1 & X_1'X_2 & X_1'X_3 \\ X_2'X_1 & X_2'X_2 & X_2'X_3 \\ X_3'X_1 & X_3'X_2 & X_3'X_3 \end{bmatrix}^{-1} \begin{bmatrix} X_1'Y_1 \\ X_2'Y_1 \\ X_3'Y_1 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{32}{47} \\ \frac{1}{94} \\ \frac{9}{47} \end{bmatrix}
\end{aligned}
$$

Estimating equation 1 using the estimates from (2) for the structural model,

$$
\begin{aligned}
\hat{\pi}_{11} &= \frac{\hat{\beta}_{11}}{1 - \hat{\gamma}_1\hat{\gamma}_2} = \frac{\frac{386}{667}}{1 - \frac{246}{667} \cdot \frac{31}{64}} \approx 0.7046 \\
\hat{\pi}_{12} &= \frac{\hat{\beta}_{22}\hat{\gamma}_1}{1 - \hat{\gamma}_1\hat{\gamma}_2} = \frac{\frac{47}{128} \cdot \frac{246}{667}}{1 - \frac{246}{667} \cdot \frac{31}{64}} \approx 0.1649 \\
\hat{\pi}_{13} &= \frac{\hat{\beta}_{32}\hat{\gamma}_1}{1 - \hat{\gamma}_1\hat{\gamma}_2} = \frac{\frac{7}{64} \cdot \frac{246}{667}}{1 - \frac{246}{667} \cdot \frac{31}{64}} \approx 0.0491
\end{aligned}
$$

Estimating equation 2 using OLS,

$$
\begin{aligned}
\begin{bmatrix} \hat{\pi}_{21} \\ \hat{\pi}_{22} \\ \hat{\pi}_{23} \end{bmatrix} &= \left(\begin{bmatrix} X_1' \\ X_2' \\ X_3' \end{bmatrix} \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} \right)^{-1} \begin{bmatrix} X_1' \\ X_2' \\ X_3' \end{bmatrix} Y_2 \\
&= \begin{bmatrix} X_1'X_1 & X_1'X_2 & X_1'X_3 \\ X_2'X_1 & X_2'X_2 & X_2'X_3 \\ X_3'X_1 & X_3'X_2 & X_3'X_3 \end{bmatrix}^{-1} \begin{bmatrix} X_1'Y_2 \\ X_2'Y_2 \\ X_3'Y_2 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} \frac{31}{94} \\ \frac{35}{94} \\ \frac{19}{94} \end{bmatrix}
\end{aligned}
$$

Estimating equation 2 using the estimates from (2) for the structural model,

$$
\begin{aligned}
\hat{\pi}_{21} &= \frac{\hat{\beta}_{11}\hat{\gamma}_2}{1 - \hat{\gamma}_1\hat{\gamma}_2} = \frac{\frac{386}{667} \cdot \frac{31}{64}}{1 - \frac{246}{667} \cdot \frac{31}{64}} \approx 0.3413 \\
\hat{\pi}_{22} &= \frac{\hat{\beta}_{22}}{1 - \hat{\gamma}_1\hat{\gamma}_2} = \frac{\frac{47}{128}}{1 - \frac{246}{667} \cdot \frac{31}{64}} \approx 0.4471 \\
\hat{\pi}_{23} &= \frac{\hat{\beta}_{32}}{1 - \hat{\gamma}_1\hat{\gamma}_2} = \frac{\frac{7}{64}}{1 - \frac{246}{667} \cdot \frac{31}{64}} \approx 0.1332
\end{aligned}
$$

Quite surprisingly, the estimates of the reduced form parameters derived from the estimations of the structural model are not very close to the OLS estimates.