Econ 203C: Systems Models

Problem Set 1

Michael Powell
Department of Economics, UCLA
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Question 1:

This question is based on the multivariate regression model of Lecture note 4. Chamberlain (1984). The model is as follows: for $i = 1, \ldots, N$,

$$\begin{aligned} y_{i1} &= \pi_1' x_i + u_{i1}, \\ \vdots & \qquad \vdots \\ y_{iT} &= \pi_T' x_i + u_{iT}, \end{aligned}$$

where $E\left[u_{it} | x_i\right] = 0$, $Cov\left(u_{it}, u_{js}\right) = \sigma_{st}$ if $i = j$ and $Cov\left(u_{it}, u_{js}\right) = 0$ if $i \neq j$, and

$$X = \begin{pmatrix} x_1' \\ \vdots \\ x_N' \end{pmatrix} = \begin{pmatrix} x_{11} & \cdots & x_{1K} \\ \vdots & \ddots & \vdots \\ x_{N1} & \cdots & x_{NK} \end{pmatrix}.$$

a. Assume that the relevant matrices are conformable and show the following Kronecker product properties:

1. $(A \otimes B)(C \otimes D) = AC \otimes BD$
2. $(A \otimes B)' = A' \otimes B'$
3. If A and B are non-singular, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

Solution:

For (1), let A be $m \times n$, C be $n \times k$, and let B and D be conformable. Then we have:

$$\begin{aligned} (A \otimes B)(C \otimes D) &= \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \begin{bmatrix} c_{11}D & \cdots & C_{1k}D \\ \vdots & \ddots & \vdots \\ c_{n1}D & \vdots & c_{nk}D \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=1}^n a_{1i}c_{i1}BD & \cdots & \sum_{i=1}^n a_{1i}c_{ik}BD \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^n a_{mi}c_{i1}BD & \cdots & \sum_{i=1}^n a_{mi}c_{ik}BD \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=1}^n a_{1i}c_{i1} & \cdots & \sum_{i=1}^n a_{1i}c_{ik} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^n a_{mi}c_{i1} & \cdots & \sum_{i=1}^n a_{mi}c_{ik} \end{bmatrix} \otimes BD \\ &= AC \otimes BD \end{aligned}$$

For (2), let A be $m \times n$ and B be $n \times k$. Then we have,

$$
\begin{aligned}
(A \otimes B) &= \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \\
&= \begin{bmatrix} a_{11}b_{11} & \cdots & a_{11}b_{1k} & & a_{1n}b_{11} & \cdots & a_{1n}b_{1k} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ a_{11}b_{n1} & \cdots & a_{11}b_{nk} & & a_{1n}b_{n1} & \cdots & a_{1n}b_{nk} \\ & \vdots & & \ddots & & \vdots & \\ a_{m1}b_{11} & \cdots & a_{m1}b_{1k} & & a_{mn}b_{11} & \cdots & a_{mn}b_{1k} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ a_{m1}b_{n1} & \cdots & a_{m1}b_{nk} & & a_{mn}b_{n1} & \cdots & a_{mn}b_{nk} \end{bmatrix}
\end{aligned}
$$

Taking the transpose:

$$
\begin{aligned}
(A \otimes B)' &= \begin{bmatrix} a_{11}b_{11} & \cdots & a_{11}b_{n1} & & a_{m1}b_{11} & \cdots & a_{m1}b_{n1} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ a_{11}b_{1k} & \cdots & a_{11}b_{nk} & & a_{m1}b_{1k} & \cdots & a_{m1}b_{nk} \\ & \vdots & & \ddots & & \vdots & \\ a_{1n}b_{11} & \cdots & a_{1n}b_{n1} & & a_{mn}b_{11} & \cdots & a_{mn}b_{n1} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ a_{1n}b_{1k} & \cdots & a_{1n}b_{nk} & & a_{mn}b_{1k} & \cdots & a_{mn}b_{nk} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}B' & \cdots & a_{m1}B' \\ \vdots & \ddots & \vdots \\ a_{1n}B' & \cdots & a_{mn}B' \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix} \otimes B' \\
&= A' \otimes B'
\end{aligned}
$$

Finally, for (3), assume that A is $n \times n$, B is $m \times m$ and both are invertible. Using the result (1) established above, we have

$$
\begin{aligned}
(A \otimes B)\left(A^{-1} \otimes B^{-1}\right) &= \left(AA^{-1} \otimes BB^{-1}\right) = (I_n \otimes I_m) = I_{nm} \\
\left(A^{-1} \otimes B^{-1}\right)(A \otimes B) &= \left(A^{-1}A \otimes B^{-1}B\right) = (I_n \otimes I_{nm}) = I_{nm}
\end{aligned}
$$

Therefore, $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

b. Using the Kronecker product notation, we can write the multivariate regression model as:

$$
y = (I \otimes X)\,\pi + u, \quad E[u|\,X] = 0, \quad Var(u|\,X) = \Sigma \otimes I_N,
$$

where Σ is a $T \times T$ matrix with the s,t elements σ_{st}.

1. Show that the GLS estimator of π is given by

$$
\hat{\pi} = \left(I \otimes (X'X)^{-1} X'\right) y.
$$

2. Show that

$$
Var(\hat{\pi}|\,X) = \Sigma \otimes (X'X)^{-1}.
$$

3. Suppose that there are linear restrictions on π of the form $\pi = G\theta$, where θ is an unrestricted parameter vector and G is a known matrix. Show that the GLS estimator for θ can be written as

$$
\hat{\theta} = \left(G'A^{-1}G\right)^{-1} G'A^{-1}\hat{\pi},
$$

where $A = Var(\hat{\pi}|\,X)$.

Solution:
Using the results from part (a), we have for the first part

$$\begin{aligned}
\hat{\pi}_{GLS} &= \left((I_N \otimes X)' (\Sigma \otimes I_N)^{-1} (I \otimes X)\right)^{-1} (I \otimes X)' (\Sigma \otimes I_N)^{-1} y \\
&= \left((I' \otimes X') \left(\Sigma^{-1} \otimes I\right) (I \otimes X)\right)^{-1} (I' \otimes X') \left(\Sigma^{-1} \otimes I\right) y \\
&= \left(I \cdot \Sigma^{-1} \cdot I \otimes X' \cdot I \cdot X\right)^{-1} \left(I' \cdot \Sigma^{-1} \otimes X' \cdot I\right) y \\
&= \left(\Sigma^{-1} \otimes X'X\right)^{-1} \left(\Sigma^{-1} \otimes X'\right) y \\
&= \left(\Sigma \otimes (X'X)^{-1}\right) \left(\Sigma^{-1} \otimes X'\right) y \\
&= \left(\Sigma \cdot \Sigma^{-1} \otimes (X'X)^{-1} X'\right) y \\
&= \left(I \otimes (X'X)^{-1} X'\right) y
\end{aligned}$$

For the second part,

$$\begin{aligned}
Var\left(\hat{\pi}_{GLS} \middle| X\right) &= Var\left(\left(I \otimes (X'X)^{-1} X'\right) y \middle| X\right) \\
&= \left(I \otimes (X'X)^{-1} X'\right) Var\left(y | X\right) \left(I \otimes (X'X)^{-1} X'\right)' \\
&= \left(I \otimes (X'X)^{-1} X'\right) (\Sigma \otimes I) \left(I \otimes X (X'X)^{-1}\right) \\
&= \left(\Sigma \otimes (X'X)^{-1} X'X (X'X)^{-1}\right) \\
&= \Sigma \otimes (X'X)^{-1}.
\end{aligned}$$

Finally, for the third part, we have that

$$y = (I \otimes X) G\theta + u$$

Which gives us that

$$Var\left(y | X\right) = Var\left(u | X\right) = \Sigma \otimes I$$

And therefore,

$$\begin{aligned}
\hat{\theta}_{GLS} &= \left(((I \otimes X) G)' (\Sigma \otimes I)^{-1} (I \otimes X) G\right)^{-1} ((I \otimes X) G)' (\Sigma \otimes I)^{-1} y \\
&= \left(G' (I \otimes X') \left(\Sigma^{-1} \otimes I\right) (I \otimes X) G\right)^{-1} G' (I \otimes X') (\Sigma \otimes I)^{-1} y \\
&= \left(G' \left(\Sigma^{-1} \otimes X'X\right) G\right)^{-1} G' \left(\Sigma^{-1} \otimes X'\right) y \\
&= \left(G' A^{-1} G\right)^{-1} G' \left(\Sigma^{-1} \otimes X'\right) y
\end{aligned}$$

Since

$$\begin{aligned}
A^{-1}\hat{\pi} &= \left(\Sigma \otimes (X'X)^{-1}\right)^{-1} \left(I \otimes (X'X)^{-1} X'\right) y \\
&= \left(\Sigma^{-1} \otimes X'X\right) \left(I \otimes (X'X)^{-1} X'\right) y \\
&= \left(\Sigma^{-1} \otimes X'X (X'X)^{-1} X'\right) y \\
&= \left(\Sigma^{-1} \otimes X'\right) y
\end{aligned}$$

We have that

$$\hat{\theta}_{GLS} = \left(G' A^{-1} G\right)^{-1} G' A^{-1} \hat{\pi}$$

Which is the desired result.

Question 2: (Question 6 in Greene, page 376):
Consider the model given by

$$\begin{aligned} y_{1i} &= \alpha_1 + \beta x_i + \varepsilon_{1i} \\ y_{2i} &= \alpha_2 + \varepsilon_{2i}, \end{aligned}$$

for $i = 1, \ldots, n$. Assume that

$$\varepsilon_i = \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \end{bmatrix} \sim D(0, \Sigma) \text{ and } \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}.$$

Prove that the GLS estimate applied to the system leads to the OLS estimates for α_1 and α_2, but to a mixture of two least-squares slopes from the regressions of y_1 on x, and y_2 on x as the estimator for β. For simplicity of the algebra, assume that the mean of the x_i's is zero, i.e., $\bar{x} = 0$.

Solution: First, I will put this model into matrix notation.

$$\begin{bmatrix} y_{1i} \\ y_{2i} \end{bmatrix} = \begin{bmatrix} 1 & x_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \end{bmatrix}$$

Stacking over all observations,

$$\begin{aligned} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} \ell & x & 0 \\ 0 & 0 & \ell \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \\ &= XB + E \end{aligned}$$

Where $y_j = \begin{bmatrix} y_{j1} \\ \vdots \\ y_{jn} \end{bmatrix}$, $\varepsilon_j = \begin{bmatrix} \varepsilon_{j1} \\ \vdots \\ \varepsilon_{jn} \end{bmatrix}$, $j = 1, 2$, $\ell = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

Here, we have that $\Sigma = Var(\varepsilon_i) = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$, where $\varepsilon_i = \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \end{bmatrix}$. The variance-covariance matrix for the regression is thus $Var(\varepsilon) = \Sigma \otimes I_n$. This gives us the following generalized least squares estimator

$$\hat{B}_{GLS} = \left(X' (\Sigma \otimes I)^{-1} X \right)^{-1} X' (\Sigma \otimes I)^{-1} y$$

$$X' \left(\Sigma^{-1} \otimes I \right) X \hat{B}_{GLS} = X' \left(\Sigma^{-1} \otimes I \right) y$$

Define the elements of Σ^{-1} as

$$\Sigma^{-1} = \begin{bmatrix} \sigma^{11} & \sigma^{12} \\ \sigma^{21} & \sigma^{22} \end{bmatrix}$$

Plugging in the relevant values, we have

$$\begin{aligned} & \begin{bmatrix} \ell' & 0 \\ x' & 0 \\ 0 & \ell' \end{bmatrix} \begin{bmatrix} \sigma^{11} I_n & \sigma^{12} I_n \\ \sigma^{21} I_n & \sigma^{22} I_n \end{bmatrix} \begin{bmatrix} \ell & x & 0 \\ 0 & 0 & \ell \end{bmatrix} \begin{bmatrix} \hat{\alpha}_{1,GLS} \\ \hat{\beta}_{GLS} \\ \hat{\alpha}_{2,GLS} \end{bmatrix} \\ = & \begin{bmatrix} \ell' & 0 \\ x' & 0 \\ 0 & \ell' \end{bmatrix} \begin{bmatrix} \sigma^{11} I_n & \sigma^{12} I_n \\ \sigma^{21} I_n & \sigma^{22} I_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \end{aligned}$$

Or

$$\begin{bmatrix} \sigma^{11}\ell' & \sigma^{12}\ell' \\ \sigma^{11}x' & \sigma^{12}x' \\ \sigma^{21}\ell' & \sigma^{22}\ell' \end{bmatrix} \begin{bmatrix} \ell\hat{\alpha}_{1,GLS} + x\hat{\beta}_{GLS} \\ \ell\hat{\alpha}_{2,GLS} \end{bmatrix} = \begin{bmatrix} \sigma^{11}\ell' & \sigma^{12}\ell' \\ \sigma^{11}x' & \sigma^{12}x' \\ \sigma^{21}\ell' & \sigma^{22}\ell' \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} \sigma^{11}\ell'\ell\hat{\alpha}_{1,GLS} + \sigma^{11}\ell'x\hat{\beta}_{GLS} + \sigma^{12}\ell'\ell\hat{\alpha}_{2,GLS} \\ \sigma^{11}x'\ell\hat{\alpha}_{1,GLS} + \sigma^{11}x'x\hat{\beta}_{GLS} + \sigma^{12}x'\ell\hat{\alpha}_{2,GLS} \\ \sigma^{21}\ell'\ell\hat{\alpha}_{1,GLS} + \sigma^{21}\ell'x\hat{\beta}_{GLS} + \sigma^{22}\ell'\ell\hat{\alpha}_{2,GLS} \end{bmatrix} = \begin{bmatrix} \sigma^{11}\ell'y_1 + \sigma^{12}\ell'y_2 \\ \sigma^{11}x'y_1 + \sigma^{12}x'y_2 \\ \sigma^{21}\ell'y_1 + \sigma^{22}\ell'y_2 \end{bmatrix}$$

Recognizing that $\ell' x = x'\ell = \sum_{i=1}^n x_i = n\bar{x} = 0$ and $\ell'\ell = n$, this reduces to

$$\begin{bmatrix} n\sigma^{11}\hat{\alpha}_{1,GLS} + n\sigma^{12}\hat{\alpha}_{2,GLS} \\ \sigma^{11}x'x\hat{\beta}_{GLS} \\ n\sigma^{21}\hat{\alpha}_{1,GLS} + n\sigma^{22}\hat{\alpha}_{2,GLS} \end{bmatrix} = \begin{bmatrix} \sigma^{11}\ell' y_1 + \sigma^{12}\ell' y_2 \\ \sigma^{11}x'y_1 + \sigma^{12}x'y_2 \\ \sigma^{21}\ell' y_1 + \sigma^{22}\ell' y_2 \end{bmatrix}$$

The second equation becomes

$$\begin{aligned} \hat{\beta}_{GLS} &= (x'x)^{-1}x'y_1 + \frac{\sigma^{12}}{\sigma^{11}}(x'x)^{-1}x'y_2 \\ &= \hat{\beta}_{1,OLS} + \frac{\sigma^{12}}{\sigma^{11}}\hat{\beta}_{2,OLS} \end{aligned}$$

The first and third equation become (recognizing that $\sigma^{12} = \sigma^{21}$)

$$\begin{bmatrix} \sigma^{11}\hat{\alpha}_{1,GLS} + \sigma^{12}\hat{\alpha}_{2,GLS} \\ \sigma^{12}\hat{\alpha}_{1,GLS} + \sigma^{22}\hat{\alpha}_{2,GLS} \end{bmatrix} = \begin{bmatrix} \sigma^{11}\bar{y}_1 + \sigma^{12}\bar{y}_2 \\ \sigma^{12}\bar{y}_1 + \sigma^{22}\bar{y}_2 \end{bmatrix}$$

Solving each equation for $\hat{\alpha}_{1,GLS}$, we get

$$\begin{aligned} \hat{\alpha}_{1,GLS} &= -\frac{\sigma^{12}}{\sigma^{11}}\hat{\alpha}_{2,GLS} + \bar{y}_1 + \frac{\sigma^{12}}{\sigma^{11}}\bar{y}_2 \\ \hat{\alpha}_{1,GLS} &= -\frac{\sigma^{22}}{\sigma^{12}}\hat{\alpha}_{2,GLS} + \bar{y}_1 + \frac{\sigma^{22}}{\sigma^{12}}\bar{y}_2 \end{aligned}$$

Or

$$\begin{aligned} -\frac{\sigma^{12}}{\sigma^{11}}\hat{\alpha}_{2,GLS} + \bar{y}_1 + \frac{\sigma^{12}}{\sigma^{11}}\bar{y}_2 &= -\frac{\sigma^{22}}{\sigma^{12}}\hat{\alpha}_{2,GLS} + \bar{y}_1 + \frac{\sigma^{22}}{\sigma^{12}}\bar{y}_2 \\ \hat{\alpha}_{2,GLS}\left(\frac{\sigma^{12}}{\sigma^{11}} - \frac{\sigma^{22}}{\sigma^{12}}\right) &= \bar{y}_2\left(\frac{\sigma^{12}}{\sigma^{11}} - \frac{\sigma^{22}}{\sigma^{12}}\right) \\ \hat{\alpha}_{2,GLS} &= \bar{y}_2 \end{aligned}$$

Which in turn gives us

$$\begin{aligned} \hat{\alpha}_{1,GLS} &= -\frac{\sigma^{22}}{\sigma^{12}}\bar{y}_2 + \bar{y}_1 + \frac{\sigma^{22}}{\sigma^{12}}\bar{y}_2 \\ &= \bar{y}_1 \end{aligned}$$

Putting this all together,

$$\begin{bmatrix} \hat{\alpha}_{1,GLS} \\ \hat{\beta}_{GLS} \\ \hat{\alpha}_{2,GLS} \end{bmatrix} = \begin{bmatrix} \bar{y}_1 \\ \hat{\beta}_{1,OLS} + \frac{\sigma^{12}}{\sigma^{11}}\hat{\beta}_{2,OLS} \\ \bar{y}_2 \end{bmatrix}$$

Recognizing that $\hat{\alpha}_{1,OLS} = \bar{y}_1 - \hat{\beta}_{1,OLS}\bar{x} = \bar{y}_1$ and $\hat{\alpha}_{2,OLS} = \bar{y}_2$, we have that

$$\begin{aligned} \hat{\alpha}_{1,GLS} &= \bar{y}_1 = \hat{\alpha}_{1,OLS} \\ \hat{\alpha}_{2,GLS} &= \bar{y}_2 = \hat{\alpha}_{2,OLS} \end{aligned}$$

And

$$\hat{\beta}_{GLS} = \theta_1\hat{\beta}_{1,OLS} + \theta_2\hat{\beta}_{2,OLS}$$

Which is the desired result.

Question 3 (Question 1 in Greene, page 422):
Consider the two equation model:

$$\begin{aligned} y_{1i} &= \gamma_1 y_{2i} + \beta_{11}x_{1i} + \beta_{21}x_{2i} + \beta_{31}x_{3i} + \varepsilon_{1i}, \\ y_{2i} &= \gamma_2 y_{1i} + \beta_{12}x_{1i} + \beta_{22}x_{2i} + \beta_{32}x_{3i} + \varepsilon_{2i}, \end{aligned}$$

where

$$\varepsilon_i = \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \end{bmatrix} \sim D(0, \Sigma) \text{ and } \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}.$$

a. Verify that neither equation is identified.
Solution:
Rewriting the above model, we have:

$$\begin{aligned} y_{1i} - \gamma_1 y_{2i} &= \beta_{11}x_{1i} + \beta_{21}x_{2i} + \beta_{31}x_{3i} + \varepsilon_{1i} \\ -\gamma_2 y_{1i} + y_{2i} &= \beta_{12}x_{1i} + \beta_{22}x_{2i} + \beta_{32}x_{3i} + \varepsilon_{2i} \end{aligned}$$

Or

$$\begin{bmatrix} y_{1i} & y_{2i} \end{bmatrix} \begin{bmatrix} 1 & -\gamma_2 \\ -\gamma_1 & 1 \end{bmatrix} = \begin{bmatrix} x_{1i} & x_{2i} & x_{3i} \end{bmatrix} \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1i} & \varepsilon_{2i} \end{bmatrix}$$

Stacking the observations, we have

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 & -\gamma_2 \\ -\gamma_1 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix} + \begin{bmatrix} \varepsilon_1 & \varepsilon_2 \end{bmatrix}$$

And if we rewrite these equations in the form

$$\begin{aligned} Y &= \begin{bmatrix} Y & X \end{bmatrix} \begin{bmatrix} I - \Gamma \\ B \end{bmatrix} + E \\ &= \begin{bmatrix} Y & X \end{bmatrix} A + E \end{aligned}$$

We have that

$$A = \begin{bmatrix} 0 & \gamma_2 \\ \gamma_1 & 0 \\ \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix}$$

Here, $m = 2$, where m is the number of endogenous variables. Let R_j be the number of restrictions in the jth column. The order conditions state that if

$$\begin{aligned} R_j &> m, \text{ equation } j \text{ is overidentified} \\ R_j &= m, \text{ equation } j \text{ is just identified} \\ R_j &< m, \text{ equation } j \text{ is underidentified} \end{aligned}$$

In this particular example, we have that $R_1 = R_2 = 1 < m$, and therefore both equations are underidentified. As a result, in neither equation can the structural parameters be identified.

b. Establish whether or not the stated restrictions are sufficient for identification (or partial identification) of the model and briefly justify your answers:

1. $\beta_{21} = \beta_{32} = 0$.
2. $\beta_{12} = \beta_{22} = 0$.
3. $\gamma_1 = 0$.

4. $\gamma_1 = \gamma_2$ and $\beta_{32} = 0$.
5. $\sigma_{12} = 0$ and $\beta_{31} = 0$.
6. $\gamma_1 = 0$ and $\sigma_{12} = 0$.
7. $\beta_{21} + \beta_{22} = 1$.
8. $\sigma_{12} = 0$ and $\beta_{21} = \beta_{22} = \beta_{31} = \beta_{32} = 0$.
9. $\sigma_{12} = 0$ and $\beta_{11} = \beta_{21} = \beta_{22} = \beta_{31} = \beta_{32} = 0$.

Solution:
1: Imposing restriction (1), we have:

$$A = \begin{bmatrix} 0 & \gamma_2 \\ \gamma_1 & 0 \\ \beta_{11} & \beta_{12} \\ 0 & \beta_{22} \\ \beta_{31} & 0 \end{bmatrix}$$

And therefore, $R_1 = R_2 = 2 = m$, which tells us that both equations are just identified.
2: Imposing restriction (2)

$$A = \begin{bmatrix} 0 & \gamma_2 \\ \gamma_1 & 0 \\ \beta_{11} & 0 \\ \beta_{21} & 0 \\ \beta_{31} & \beta_{32} \end{bmatrix}$$

Which gives us that $R_1 = 1 < m$, so that equation 1 is underidentified, and $R_2 = 3 > m$, so that equation 2 is overidentified.
3: Imposing restriction (3)

$$A = \begin{bmatrix} 0 & \gamma_2 \\ 0 & 0 \\ \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix}$$

Here, we have that $R_1 = 2 = m$, so equation 1 is just identified. $R_2 = 1 < m$, and we have that equation 2 is underidentified.
4: Restriction (4) is where things start to get interesting. Imposing restriction (4), we have

$$A = \begin{bmatrix} 0 & \gamma_2 \\ \gamma_2 & 0 \\ \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & 0 \end{bmatrix}$$

Since $R_2 = 2 = m$, we have that equation (2) is just identified. In particular, we can estimate γ_2, so in equation (1), we can treat it as fixed (and hence as an additional restriction). This gives us that $R_1 = 2 = m$ and therefore equation (1) is just identified as well.
5: For restriction (5), first note that

$$A = \begin{bmatrix} 0 & \gamma_2 \\ \gamma_1 & 0 \\ \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ 0 & \beta_{32} \end{bmatrix}$$

This gives us that $R_1 = 2 = m$ or that equation (1) is just identified. Recall that we can express the variance covariance matrix for the system of equations $\Sigma = \Gamma'\Omega\Gamma$, where Ω is the variance-covariance matrix for the reduced-form model (and is therefore identified). The derivation for this expression is as follows. For $i = 1, \ldots, n$, we have that

$$\underset{1\times m}{y_i}\underset{m\times m}{\Gamma} = \underset{1\times k}{x_i}\underset{k\times m}{B} + \underset{1\times m}{\varepsilon_i} \tag{1}$$

Where ε_i are assumed to be i.i.d. with $E\left[\varepsilon_i | X\right] = 0$. The variance covariance matrix for the structural model is therefore

$$\Sigma = Var\left(y_i\Gamma | X\right) = Var\left(\varepsilon_i | X\right) = E\left[\varepsilon_i'\varepsilon_i | X\right]$$

If we assume that Γ is invertible, we can postmultiply expression (1) above by Γ^{-1} to obtain

$$y_i = x_i B\Gamma^{-1} + v_i$$

Where $v_i = \varepsilon_i\Gamma^{-1}$. The variance covariance matrix for the reduced form model is therefore

$$\begin{aligned}
\Omega &= Var\left(y_i | X\right) = Var\left(v_i | X\right) = E\left[v_i'v_i | X\right] \\
&= E\left[\left(\Gamma^{-1}\right)'\varepsilon_i'\varepsilon_i\Gamma^{-1}\middle| X\right] \\
&= \left(\Gamma^{-1}\right)' E\left[\varepsilon_i'\varepsilon_i | X\right]\Gamma^{-1} \\
&= \left(\Gamma^{-1}\right)'\Sigma\Gamma^{-1}
\end{aligned}$$

Or, solving for Σ, we have that $\Sigma = \Gamma'\Omega\Gamma$. Adapting this to this particular example, we have the following:

$$\begin{aligned}
\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} &= \begin{bmatrix} 1 & -\gamma_1 \\ -\gamma_2 & 1 \end{bmatrix}\begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix}\begin{bmatrix} 1 & -\gamma_2 \\ -\gamma_1 & 1 \end{bmatrix} \\
\begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{bmatrix} &= \begin{bmatrix} \omega_{11} - \gamma_1\omega_{12} & \omega_{12} - \gamma_1\omega_{22} \\ -\gamma_2\omega_{11} + \omega_{12} & -\gamma_2\omega_{12} + \omega_{22} \end{bmatrix}\begin{bmatrix} 1 & -\gamma_2 \\ -\gamma_1 & 1 \end{bmatrix} \\
\begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{bmatrix} &= \begin{bmatrix} \omega_{11} - \gamma_1\omega_{12} - \gamma_1\omega_{12} + \gamma_1^2\omega_{22} & -\gamma_2\omega_{11} + \gamma_1\gamma_2\omega_{12} + \omega_{12} - \gamma_1\omega_{22} \\ -\gamma_2\omega_{11} + \omega_{12} + \gamma_1\gamma_2\omega_{12} - \gamma_1\omega_{22} & \gamma_2^2\omega_{11} - \gamma_2\omega_{12} - \gamma_2\omega_{12} + \omega_{22} \end{bmatrix} \\
\begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{bmatrix} &= \begin{bmatrix} \omega_{11} - 2\gamma_1\omega_{12} + \gamma_1^2\omega_{22} & -\gamma_2\omega_{11} + \left(\gamma_1\gamma_2 + 1\right)\omega_{12} - \gamma_1\omega_{22} \\ -\gamma_2\omega_{11} + \left(\gamma_1\gamma_2 + 1\right)\omega_{12} - \gamma_1\omega_{22} & \gamma_2^2\omega_{11} - 2\gamma_2\omega_{12} + \omega_{22} \end{bmatrix}
\end{aligned}$$

Where $\omega_{11}, \omega_{12}, \omega_{22}$ are identified. Since equation (1) is identified, we can estimate γ_1 and therefore treat it as a given. Therefore, we have that

$$0 = -\gamma_2\omega_{11} + \left(\gamma_1\gamma_2 + 1\right)\omega_{12} - \gamma_1\omega_{22}$$

Or

$$\begin{aligned}
\gamma_2\omega_{11} - \gamma_1\gamma_2\omega_{12} &= \omega_{12} - \gamma_1\omega_{22} \\
\gamma_2\left(\omega_{11} - \gamma_1\omega_{12}\right) &= \omega_{12} - \gamma_1\omega_{22} \\
\gamma_2 &= \frac{\omega_{12} - \gamma_1\omega_{22}}{\omega_{11} - \gamma_1\omega_{12}}
\end{aligned}$$

In other words, γ_2 can be expressed as a function of parameters which are identified. As a result, we can take γ_2 as a given. This gives us that $R_2 = 2 = m$, or that equation 2 is just identified.

6: At first glance, imposing restriction (6) gives us

$$A = \begin{bmatrix} 0 & \gamma_2 \\ 0 & 0 \\ \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix}$$

In particular, $R_1 = 2 = m$, which tells us that the first equation is just identified. As for the second equation, using the derivation from above, we have that

$$\begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \omega_{11} - 2\gamma_1\omega_{12} & -\gamma_2\omega_{11} + \omega_{12} \\ -\gamma_2\omega_{11} + \omega_{12} & \gamma_2^2\omega_{11} - 2\gamma_2\omega_{12} + \omega_{22} \end{bmatrix}$$

In particular,

$$\begin{aligned} -\gamma_2\omega_{11} + \omega_{12} &= 0 \\ \gamma_2 &= \frac{\omega_{12}}{\omega_{11}} \end{aligned}$$

Or γ_2 can be expressed as a function of parameters which are identified, and therefore, γ_2 is identified and can thus be treated as a constant. This gives us that $R_2 = 2 = m$ or that equation 2 is just identified.

7: Here, I will make use of the relationship $\Pi = B\Gamma^{-1}$ where Π is completely identified, to see if we can identify the equations. To derive this relationship, recall that we can write the model as

$$y_i\Gamma = x_iB + \varepsilon_i,\ i = 1, \ldots, n$$

Assuming Γ is invertible, we have

$$\begin{aligned} y_i &= x_iB\Gamma^{-1} + v_i,\ i = 1, \ldots, n \\ &= x_i\Pi + v_i \end{aligned}$$

Using standard OLS, we can estimate $\Pi = B\Gamma^{-1}$ or $B = \Pi\Gamma$. Adapting this expression to this particular model, we have

$$\begin{aligned} \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & 1 - \beta_{21} \\ \beta_{31} & \beta_{32} \end{bmatrix} &= \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \\ \pi_{31} & \pi_{32} \end{bmatrix} \begin{bmatrix} 1 & -\gamma_2 \\ -\gamma_1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \pi_{11} - \pi_{12}\gamma_1 & \pi_{12} - \pi_{11}\gamma_2 \\ \pi_{21} - \pi_{22}\gamma_1 & \pi_{22} - \pi_{21}\gamma_2 \\ \pi_{31} - \pi_{32}\gamma_1 & \pi_{32} - \pi_{31}\gamma_2 \end{bmatrix} \end{aligned}$$

Can we hope to identify β_{21} in this model?

$$\begin{aligned} \beta_{21} &= \pi_{21} - \pi_{22}\gamma_1 \\ 1 - \beta_{21} &= \pi_{22} - \pi_{21}\gamma_2 \end{aligned}$$

This is two equations in three unknowns $(\beta_{21}, \gamma_1, \gamma_2)$, which does not have a solution. Therefore, this restriction does not help us identify either equation, which gives us that we still have $R_1 = R_2 = 1 < m$ and both equations are underidentified.

8: If we impose the restriction $\sigma_{12} = 0$ and $\beta_{21} = \beta_{22} = \beta_{31} = \beta_{32} = 0$, the model becomes

$$\begin{aligned} y_{1i} - \gamma_1 y_{2i} &= \beta_{11}x_{1i} + \varepsilon_{1i} \\ -\gamma_2 y_{1i} + y_{2i} &= \beta_{12}x_{1i} + \varepsilon_{2i} \end{aligned}$$

If we write this model in the form $Y = \begin{bmatrix} Y & X \end{bmatrix} A + E$, we have that

$$A = \begin{bmatrix} 0 & \gamma_2 \\ \gamma_1 & 0 \\ \beta_{11} & \beta_{12} \end{bmatrix}$$

Can we hope to identify this model by imposing the $\sigma_{12} = 0$ restriction? From the derivation above, we have that

$$\begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \omega_{11} - 2\gamma_1\omega_{12} + \gamma_1^2\omega_{22} & -\gamma_2\omega_{11} + (\gamma_1\gamma_2 + 1)\,\omega_{12} - \gamma_1\omega_{22} \\ -\gamma_2\omega_{11} + (\gamma_1\gamma_2 + 1)\,\omega_{12} - \gamma_1\omega_{22} & \gamma_2^2\omega_{11} - 2\gamma_2\omega_{12} + \omega_{22} \end{bmatrix}$$

Here, we have three equations in four unknowns $(\sigma_{11}, \sigma_{22}, \gamma_1, \gamma_2)$:

$$\begin{aligned}
\sigma_{11} &= \omega_{11} - 2\gamma_1\omega_{12} + \gamma_1^2\omega_{22} \\
0 &= -\gamma_2\omega_{11} + (\gamma_1\gamma_2 + 1)\,\omega_{12} - \gamma_1\omega_{22} \\
\sigma_{22} &= \gamma_2^2\omega_{11} - 2\gamma_2\omega_{12} + \omega_{22}
\end{aligned}$$

This system of equations does not have a solution. Since $R_1 = R_2 = 1 < m$, we conclude that both equation 1 and equation 2 are underidentified.

9: Imposing the restriction $\beta_{11} = \beta_{21} = \beta_{22} = \beta_{31} = \beta_{32} = 0$, the model becomes

$$\begin{aligned}
y_{1i} - \gamma_1 y_{2i} &= \varepsilon_{1i} \\
-\gamma_2 y_{1i} + y_{2i} &= \beta_{12}x_{1i} + \varepsilon_{2i}
\end{aligned}$$

Writing this model in the form $Y = \begin{bmatrix} Y & X \end{bmatrix} A + E$, we see that

$$A = \begin{bmatrix} 0 & \gamma_2 \\ \gamma_1 & 0 \\ 0 & \beta_{12} \end{bmatrix}$$

And see that $R_1 = 2 = m$. Therefore, equation 1 is just identified, and we can estimate γ_1. Next, using the restriction $\sigma_{12} = 0$, we see that

$$0 = -\gamma_2\omega_{11} + (\gamma_1\gamma_2 + 1)\,\omega_{12} - \gamma_1\omega_{22}$$

$$\begin{aligned}
\gamma_2\left(\omega_{11} - \gamma_1\omega_{12}\right) &= \omega_{12} - \gamma_1\omega_{22} \\
\gamma_2 &= \frac{\omega_{12} - \gamma_1\omega_{22}}{\omega_{11} - \gamma_1\omega_{12}}
\end{aligned}$$

Which is a function of parameters which can be estimated. Therefore, γ_2 is identified, and we can treat it as a constant, giving us that $R_2 = 2 = m$, from which we conclude that equation 2 is just identified.

Question 4 (Question 3 in Greene, page 423):
Check the identifiability of the parameters of the following model:

$$\begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{bmatrix} 1 & \gamma_{12} & 0 & 0 \\ \gamma_{21} & 1 & \gamma_{23} & \gamma_{24} \\ 0 & \gamma_{32} & 1 & \gamma_{34} \\ \gamma_{41} & \gamma_{42} & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix} \begin{bmatrix} 0 & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & 1 & 0 & \beta_{24} \\ \beta_{31} & \beta_{32} & \beta_{33} & 0 \\ 0 & 0 & \beta_{43} & \beta_{44} \\ 0 & \beta_{52} & 0 & 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \end{bmatrix}.$$

Solution:
Rewriting the above equation in the form

$$\begin{aligned} Y &= \begin{bmatrix} Y & X \end{bmatrix} \begin{bmatrix} I - \Gamma \\ B \end{bmatrix} + E \\ &= \begin{bmatrix} Y & X \end{bmatrix} A + E \end{aligned}$$

We have that

$$A = \begin{bmatrix} 0 & -\gamma_{12} & 0 & 0 \\ -\gamma_{21} & 0 & -\gamma_{23} & -\gamma_{24} \\ 0 & -\gamma_{32} & 0 & -\gamma_{34} \\ -\gamma_{41} & -\gamma_{42} & 0 & 0 \\ 0 & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & 1 & 0 & \beta_{24} \\ \beta_{31} & \beta_{32} & \beta_{33} & 0 \\ 0 & 0 & \beta_{43} & \beta_{44} \\ 0 & \beta_{52} & 0 & 0 \end{bmatrix}$$

Here, we have that $m = 4$, where m is the column/row dimension of Γ. Let R_j denote the number of restrictions imposed in the jth column. The order conditions state that if

$$\begin{aligned} R_j &> m, \text{ equation } j \text{ is overidentified} \\ R_j &= m, \text{ equation } j \text{ is just identified} \\ R_j &< m, \text{ equation } j \text{ is underidentified} \end{aligned}$$

A quick glance at the matrix A shows us that

$$\begin{aligned} R_1 &= 5 > m, \text{ equation 1 is overidentified} \\ R_2 &= 3 < m, \text{ equation 2 is underidentified} \\ R_3 &= 5 > m, \text{ equation 3 is overidentified} \\ R_4 &= 4 = m, \text{ equation 4 is just identified} \end{aligned}$$

Therefore, we can identify all the structural parameters in equations 1, 3, and 4, but we cannot do so in equation 2.