Econ 203C: Systems Models

Problem Set 1

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Question 1:
Consider the simple consumption model given by

$$c_i = \beta_1 + \beta_2 y_i^* + u_i,\ u_i \overset{iid}{\sim} D\left(0, \sigma^2\right),$$

where c_i is the logarithm of consumption by hoseuhold i, and y_i^* is the permanent income of that household, which is not observed. Instead, we only observe the current household income y_i

$$y_i = y_i^* + v_i$$

where $v_i \overset{iid}{\sim} D\left(0, \omega^2\right)$, is assumed to be uncorrelated with y_i^* and u_i. Consider now the regression

$$c_i = \beta_1 + \beta_2 y_i + \varepsilon_i$$

Under some assumption it is reasonable to believe that the true parameter β_{20} is positive.
a. Show that y_i is negatively correlated with ε_i.
Solution: First, it is important to notice that $Corr\left(y_i, \varepsilon_i\right) < 0$ if and only if $E\left[y_i \varepsilon_i\right] < 0$:

$$\begin{aligned} sgn\left[Corr\left(y_i, \varepsilon_i\right)\right] &= sgn\left[\frac{Cov\left(y_i, \varepsilon_i\right)}{\underbrace{\sqrt{Var\left(y_i\right)}}_{>0}\underbrace{\sqrt{Var\left(\varepsilon_i\right)}}_{>0}}\right] \\ &= sgn\left(Cov\left(y_i, \varepsilon_i\right)\right) \end{aligned}$$

But

$$\begin{aligned} Cov\left(y_i, \varepsilon_i\right) &= E\left[y_i \varepsilon_i\right] - E\left[y_i\right] \underbrace{E\left[\varepsilon_i\right]}_{=0} \\ &= E\left[y_i \varepsilon_i\right] \end{aligned}$$

Thus,

$$sgn\left[Corr\left(y_i, \varepsilon_i\right)\right] = sgn\left[E\left[y_i \varepsilon_i\right]\right]$$

Where $sgn\left(x\right) \equiv 1_{\{x \geq 0\}}$. Thus, it suffices to show that $E\left[y_i \varepsilon_i\right] < 0$. For this, I will strengthen one of the initial assumptions of this problem, since otherwise, the claim we are trying to show simply is not true: $u_i |\, y_i^* \overset{iid}{\sim} D\left(0, \sigma^2\right)$ which implies that $E\left[y_i^* u_i\right] = 0$.

$$\begin{aligned} E\left[y_i \varepsilon_i\right] &= E\left[\left(y_i^* + v_i\right)\left(c_i - \beta_1 - \beta_2 y_i\right)\right] \\ &= E\left[\left(y_i^* + v_i\right)\left(c_i - \beta_1 - \beta_2 y_i^* - \beta_2 v_i\right)\right] \\ &= E\left[\left(y_i^* + v_i\right)\left(u_i - \beta_2 v_i\right)\right] \\ &= \underbrace{E\left[y_i^* u_i\right]}_{=0} + \underbrace{E\left[v_i u_i\right]}_{=0} - \beta_2 \underbrace{E\left[y_i^* v_i\right]}_{=0} - \beta_2 \underbrace{E\left[v_i^2\right]}_{=\omega^2} \\ &= -\beta_2 \omega^2 < 0 \end{aligned}$$

Since $\beta_2 > 0$ and $\omega^2 > 0$.

b. Using the result in (a), compute the probaiblity limit of the OLS estimator for β_{20}, say $\hat{\beta}_2$, from the second regression. Show that this probability limit is less than β_{20}.

Solution: The OLS estimator is given by the partitioned regression formula

$$\begin{aligned}
\hat{\beta}_2 &= \left(y'M^0y\right)^{-1} y'M^0c \\
&= \left(y'M^0y\right)^{-1} y'M^0\left(i\beta_1 + y\beta_2 + \varepsilon\right) \\
&= \beta_2 + \left(y'M^0y\right)^{-1} y'M^0\varepsilon
\end{aligned}$$

Where I used the fact that $M^0 i \equiv \left(I - i\left(i'i\right)^{-1} i'\right) i = i - i\left(i'i\right)^{-1} i'i = i - i = 0$. This gives us:

$$\hat{\beta}_2 = \beta_2 + \left(\frac{1}{n}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2\right)^{-1} \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)\varepsilon_i$$

Assuming that $y_i \neq y_j$ for at least some $i \neq j$ (so that $y_i - \bar{y}$ is not identically equal to zero), we then have that (since y_i, ε_i are *iid*)

$$p\lim \hat{\beta}_2 = \beta_2 + \left(E\left[\left(y_i - E\left[y_i\right]\right)^2\right]\right)^{-1} E\left[\left(y_i - E\left[y_i\right]\right)\varepsilon_i\right]$$

By the weak law of large numbers for *iid* data, Mann-Wald's theorem, and Slutsky's theorem. Next, note that

$$\begin{aligned}
E\left[\left(y_i - E\left[y_i\right]\right)\varepsilon_i\right] &= E\left[y_i\varepsilon_i - E\left[y_i\right]\varepsilon_i\right] \\
&= E\left[y_i\varepsilon_i\right] - E\left[y_i\right]\underbrace{E\left[\varepsilon_i\right]}_{=0} \\
&= E\left[y_i\varepsilon_i\right]
\end{aligned}$$

This gives us:

$$\begin{aligned}
p\lim \hat{\beta}_2 &= \beta_2 + \underbrace{\left(Var\left(y_i\right)\right)^{-1}}_{>0}\underbrace{E\left[y_i\varepsilon_i\right]}_{<0} \\
&< \beta_2
\end{aligned}$$

Where I used the result from part (a) that $E\left[y_i\varepsilon_i\right] < 0$.

Question 2:
Suppose that the data generating process that simultaneously determine x and y is given by

$$\begin{aligned} y &= x\beta_0 + \sigma_u u, \\ x &= w\pi_0 + \sigma_v v, \end{aligned}$$

where y, x, u and v are all $n \times 1$ vectors, u and v have standard normal distribution and $E[u_i v_i] = \rho$. Suppose also that there exist an instrument for x, say w. Assume $w'w = 1$.

a. Show that $E[u_i|\, v_i] = \rho v_i$, so we can write $u_i = \rho v_i + \varepsilon_i$, $E[\varepsilon_i|\, v_i] = 0$.

Solution: Recall that if

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \begin{bmatrix} \sigma_{XX} & \sigma_{XY} \\ \sigma_{XY} & \sigma_{YY} \end{bmatrix} \right)$$

Then

$$X|\, Y \sim N\left(\mu_X + \frac{\sigma_{XY}}{\sigma_{YY}} (Y - \mu_Y), \sigma_{XX} - \frac{\sigma_{XY}}{\sigma_{YY}} \right)$$

And thus, $E[X\ Y|] = \mu_X + \frac{\sigma_{XY}}{\sigma_{YY}} (Y - \mu_Y)$. Here, we have that

$$\begin{bmatrix} u_i \\ v_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right)$$

Therefore, $E[u_i|\, v_i] = 0 + \frac{\rho}{1}(v_i - 0) = \rho v_i$. This gives us the desired result that

$$u_i = \rho v_i + \varepsilon_i, \text{ where } E[\varepsilon_i|\, v_i] = 0.$$

b. Define the instrumental variable β_0, say $\hat{\beta}_{IV}$.

Solution: Since w is a valid instrument for x define $\hat{\beta}_{IV}$ to be the instrumental variable estimator for β_0 as

$$\hat{\beta}_{IV} = (w'x)^{-1} w'y$$

c. Show that for the estimator defined in (b)

$$\hat{\beta}_{IV} - \beta_0 = \frac{\sigma_u w' (\rho v + \varepsilon)}{\pi_0 + \sigma_v w'v},$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)'$.

Solution: Substituting in $y = x\beta_0 + \sigma_u u$ into the expression from part (b), we have:

$$\begin{aligned} \hat{\beta}_{IV} &= (w'x)^{-1} w' (x\beta_0 + \sigma_u u) \\ &= \beta_0 + \sigma_u \underbrace{(w'x)}_{1\times 1}{}^{-1} w'u \end{aligned}$$

Substituting in the result that $u = \rho v + \varepsilon$ from part (a) and $x = w\pi_0 + \sigma_v v$, we get:

$$\begin{aligned} \hat{\beta}_{IV} - \beta_0 &= \frac{\sigma_u w' (\rho v + \varepsilon)}{w' (w\pi_0 + \sigma_v v)} \\ &= \frac{\sigma_u w' (\rho v + \varepsilon)}{w'w\pi_0 + \sigma_v w'v} \\ &= \frac{\sigma_u w' (\rho v + \varepsilon)}{\pi_0 + \sigma_v w'v} \end{aligned}$$

Since $w'w = 1$. This is the desired result.

d. Show that if $\sigma_v = 0$, then the estimator defined in (b) is an unbiased estimator for β_0. Interpret this result.

Solution: If $\sigma_v = 0$, then the result from part (c) reduces to

$$\begin{aligned}
\hat{\beta}_{IV} &= \beta_0 + \frac{\sigma_u w' (\rho v + \varepsilon)}{\pi_0} = \beta_0 + \frac{\sigma_u \rho w' v + \sigma_u w' \varepsilon}{\pi_0} \\
&= \beta_0 + \frac{\sigma_u \rho w' v + \sigma_u w' (u - \rho v)}{\pi_0} \\
&= \beta_0 + \frac{\sigma_u \rho w' v + \sigma_u w' u - \sigma_u \rho w' v}{\pi_0} \\
&= \beta_0 + \frac{\sigma_u w' u}{\pi_0}
\end{aligned}$$

Taking expectations, we see:

$$E\left[\hat{\beta}_{IV}\right] = \beta_0 + \frac{\sigma_u}{\pi_0} \underbrace{E\left[w'u\right]}_{=0} = \beta_0$$

Since w is a valid instrument for x.

How do we interpret this result? If $\sigma_v = 0$, then we have the identity that $x = w\pi_0$. In order for $E\left[w'u\right] = 0$, it must be that $\frac{1}{\pi_0} E\left[x'u\right] = 0$, or $E\left[x'u\right] = 0$. That is, there is no first-order deviation from the NCRM to begin with, and $\hat{\beta}_{IV}$ is nothing more than the standard OLS estimator, which we already know is unbiased.

Question 3:
Consider the regression model given by

$$y_i = x_i'\beta_0 + u_i,\ i = 1, \ldots, n$$

where x_i is a $k \times 1$ vector of regressors and β_0 is a $k \times 1$ vector of unknown parameters. Assume that $u_i \overset{iid}{\sim} D\left(0, \sigma_u^2\right)$, but $E\left[x_i u_i\right] \neq 0$. Let z_i be an $l \times 1$ vector, with $l > k$.

a. What would constitute z_i as an instrument for x_i? Discuss briefly.

Solution: A random vector z_i is an instrument for x_i if the following two conditions are satisfied:

1) $p\lim \frac{1}{n} \sum_{i=1}^{n} z_i x_i'$ is non-singular. For the *iid* case, this reduces to the assumption that $E\left[z_i x_i'\right]$ is nonsingular.

2) $p\lim \frac{1}{n} \sum_{i=1}^{n} z_i u_i = 0$. For the *iid* case, this reduces to the assumption that $E\left[z_i u_i\right] = 0$.

That is, z_i is an instrument for x_i if z_i is, in some sense, correlated to x_i and is uncorrelated to u_i.

b. Suppose we define

$$\hat{\beta}_{IV} = \left(X'ZA^{-1}Z'X\right)^{-1} X'ZA^{-1}Z'y$$

where X is an $n \times k$ matrix with rows given by x_i', y is an $n \times 1$ vector of the stacked y_i, Z is an $n \times l$ matrix with rows given by z_i', and A is an $n \times n$ non-stochastic, non-singular matrix. Show that $\hat{\beta}_{IV}$ is a consistent estimator for β_0.

Solution: We can rewrite $\hat{\beta}_{IV}$ as follows:

$$\begin{aligned} \hat{\beta}_{IV} &= \left(X'ZA^{-1}Z'X\right)^{-1} X'ZA^{-1}Z'\left(X\beta_0 + u\right) \\ &= \beta_0 + \left(X'ZA^{-1}Z'X\right)^{-1} X'ZA^{-1}Z'u \end{aligned}$$

$$\hat{\beta}_{IV} = \beta_0 + \left(\left(\frac{1}{n}\sum_{i=1}^{n} z_i x_i'\right)' A^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} z_i x_i'\right)\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} z_i x_i'\right)' A^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} z_i u_i\right)$$

Here, we have that since z_i is an instrument for x_i by the weak law of large numbers, $p\lim \frac{1}{n} \sum_{i=1}^{n} z_i x_i' = E\left[z_i x_i'\right]$ is nonsingular and $p\lim \frac{1}{n} \sum_{i=1}^{n} z_i u_i = E\left[z_i u_i\right] = 0$. Thus, by Slutsky's theorem and the Mann-Wald theorem, we have:

$$\begin{aligned} p\lim \hat{\beta}_{IV} &= \beta_0 + \left(E\left[z_i x_i'\right]' A^{-1} E\left[z_i x_i'\right]\right)^{-1} E\left[z_i x_i'\right]' A^{-1} \underbrace{E\left[z_i u_i\right]}_{=0} \\ &= \beta_0 \end{aligned}$$

And therefore, $\hat{\beta}_{IV}$ is consistent for β_0.

c. Show that, under some regularity conditions,

$$\sqrt{n}\left(\hat{\beta}_{IV} - \beta_0\right) \xrightarrow{d} N\left(0, \Lambda\right)$$

and provide Λ.

Solution: We can rearrange the above expression to get:

$$\sqrt{n}\left(\hat{\beta}_{IV} - \beta_0\right) = \left(\left(\frac{1}{n}\sum_{i=1}^{n} z_i x_i'\right)' A^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} z_i x_i'\right)\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} z_i x_i'\right)' A^{-1} \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} z_i u_i\right)$$

By the central limit theorem, we have that

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} z_i u_i\right) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} z_i u_i - E\left[z_i u_i\right]\right) \xrightarrow{d} N\left(0, Var\left(z_i u_i\right)\right)$$

And

$$\begin{aligned} Var\left(z_i u_i\right) &= E\left[\left(z_i u_i\right)\left(z_i u_i\right)'\right] - \underbrace{E\left[z_i u_i\right]}_{=0} E\left[z_i u_i\right]' \\ &= E\left[z_i u_i u_i z_i'\right] = E\left[z_i z_i' u_i^2\right] \end{aligned}$$

Thus, by Slutsky's theorem, the Mann-Wald theorem, and the weak law of large numbers for *iid* data, we have:

$$\sqrt{n}\left(\hat{\beta}_{IV} - \beta_0\right) \xrightarrow{d} \left(E\left[z_i x_i'\right]' A^{-1} E\left[z_i x_i'\right]\right)^{-1} E\left[z_i x_i'\right]' A^{-1} N X$$

Where $X \sim N\left(0, E\left[z_i z_i' u_i^2\right]\right)$. Therefore, we have that

$$\sqrt{n}\left(\hat{\beta}_{IV} - \beta_0\right) \xrightarrow{d} N\left(0, \Lambda\right)$$

Where

$$\Lambda = \left(E\left[z_i x_i'\right]' A^{-1} E\left[z_i x_i'\right]\right)^{-1} E\left[z_i x_i'\right]' A^{-1} E\left[z_i z_i' u_i^2\right] \left(A^{-1}\right)' E\left[z_i x_i'\right] \left(E\left[z_i x_i'\right]' \left(A^{-1}\right)' E\left[z_i x_i'\right]\right)$$