Econ 203C: Systems Models

Practice Questions

Michael Powell
Department of Economics, UCLA
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Question 1:

Suppose that you are given the random sample $y_1, \ldots, y_n$ from a distribution whose density function is given by

$$f(y) = \frac{2}{\theta^2}(\theta - y), \text{ for } 0 < y < \theta.$$

(1) Provide the MLE for θ, say $\hat{\theta}_n$.

Solution: Here, we have that $f_{Y_i}(y_i) = \frac{2}{\theta^2}(\theta - y_i)$ for $0 < y_i < \theta$. The likelihood function is thus given by

$$\begin{aligned} L_n(\theta; \{Y_i\}) &= \prod_{i=1}^n f_{Y_i}(Y_i) \\ &= \prod_{i=1}^n \frac{2}{\theta^2}(\theta - Y_i) 1_{\{0<Y_i<\theta\}} \\ &= \left(\frac{2}{\theta^2}\right)^n \prod_{i=1}^n (\theta - Y_i) 1_{\{0<Y_i<\theta\}} \end{aligned}$$

Taking logs,

$$\ln L_n(\theta; \{Y_i\}) = n \log 2 - 2n \log \theta + \sum_{i=1}^n \log(\theta - Y_i) + \sum_{i=1}^n \log 1_{\{0<Y_i<\theta\}}$$

The first order conditions are thus

$$(\theta) : -\frac{2n}{\hat{\theta}_n^{ML}} + \sum_{i=1}^n \frac{1}{\hat{\theta}_n^{ML} - Y_i} = 0$$

Or

$$\begin{aligned} \sum_{i=1}^n \frac{1}{\hat{\theta}_n^{ML} - Y_i} &= \frac{2n}{\hat{\theta}_n^{ML}} \\ \frac{n\hat{\theta}_n^{ML} - \sum_{i=1}^n Y_i}{\prod_{i=1}^n \left(\hat{\theta}_n^{ML} - Y_i\right)} &= \frac{2n}{\hat{\theta}_n^{ML}} \end{aligned}$$

$$n\left(\hat{\theta}_n^{ML}\right)^2 - \hat{\theta}_n^{ML}\left(\sum_{i=1}^n Y_i\right) = 2n \prod_{i=1}^n \left(\hat{\theta}_n^{ML} - Y_i\right)$$

Solving explicitly for $\hat{\theta}_n^{ML}$ requires finding the roots of an n^{th} degree polynomial which, for $n \geq 5$, is generically impossible by the Abel-Ruffini theorem, so I will just say that $\hat{\theta}_n^{ML}$ is implicitly defined as a solution to the above equation.

(2) Provide the asymptotic distribution for $\hat{\theta}_n$.

Solution: Recall that for an ML estimator,

$$\sqrt{n}\left(\hat{\theta}_n^{ML} - \theta_0\right) \xrightarrow{d} N\left(0, I_1(\theta_0)^{-1}\right)$$

Where

$$I_1(\theta_0) = E\left[-\frac{\partial^2 \ln f_{Y_i}(Y_i)}{\partial \theta^2}\right]$$

Taking these derivatives, we have

$$\begin{aligned}
\frac{\partial \ln f_{Y_i}(Y_i)}{\partial \theta} &= -\frac{2}{\theta} + \frac{1}{\theta - Y_i} \\
\frac{\partial^2 \ln f_{Y_i}(Y_i)}{\partial \theta^2} &= \frac{2}{\theta^2} - \frac{1}{(\theta - Y_i)^2}
\end{aligned}$$

Which gives us

$$I_1(\theta_0) = E\left[\frac{1}{(\theta_0 - Y_i)^2} - \frac{2}{\theta_0^2}\right]$$

Remark 1 *It can be shown that this expectation does not exist.*

Thus, the asymptotic distribution for $\hat{\theta}_n$ is given by

$$I_1(\theta_0)^{-1} = \frac{1}{E\left[\frac{1}{(\theta_0 - Y_i)^2} - \frac{2}{\theta_0^2}\right]}$$

(3) Is the estimator in (1) an unbiased estimator for θ? Justify your answer.
Solution:

(4) Provide a test statistic for testing the null hypothesis $H_0 : \theta^2 = 1$ against the hypothesis $H_1 : \theta^2 < 1$.

Question 2:
Consider the following two equations:

$$E\left[y_i|\,x_i\right] = x_i'\beta \tag{3}$$
$$E^*\left[y_i|\,x_i\right] = x_i'\beta^\ell, \tag{4}$$

for $i = 1, \ldots, n$. Note that equation (3) states that the true conditional expectation of y_i, conditional on x_i, is linear, while equation (4) characterizes the linear *predictor* of y_i conditional on x_i. Let $\hat{\beta}$ and $\hat{\beta}^\ell$, be the least-squares estimators of β and β^ℓ, respectively. Define

$$\alpha_n = \sqrt{n}\left(\hat{\beta} - \beta\right), \text{ and}$$
$$\alpha_n^\ell = \sqrt{n}\left(\hat{\beta}^\ell - \beta^\ell\right).$$

(1) Derive the asymptotic distribution of α_n.
(2) Derive the asymptotic distribution of α_n^ℓ.
(3) Under what conditions will α_n and α_n^ℓ have the same asymptotic distribution?

Question 3:
Consider the linear model

$$y_i = x_i'\beta + u_i,$$

where

$$\begin{aligned} u_i &= \sigma(x_i)\,\varepsilon_i, \\ \sigma(x_i) &= 1 + x_i'\gamma, \text{ and} \end{aligned}$$

$$\varepsilon_i|\, x_i \overset{iid}{\sim} N\left(0, \sigma_\varepsilon^2\right).$$

(1) Provide an estimator for β and show that the suggested estimator is consistent.
Solution: Let

$$\begin{aligned} \hat{\beta}_n^{OLS} &= \left(X'X\right)^{-1} X'Y \\ &= \left(X'X\right)^{-1} X'\left(X\beta + u\right) \\ &= \beta + \left(X'X\right)^{-1} X'u \\ &= \beta + \left(\frac{1}{n}\sum_{i=1}^{n} X_iX_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_iu_i \end{aligned}$$

Assuming the data are i.i.d. and that $E\left[|X_iX_i'|\right] < +\infty$, $\frac{1}{n}\sum_{i=1}^{n} X_iX_i' \xrightarrow{p} E\left[X_iX_i'\right]$. By the continuity theorem, $\left(\frac{1}{n}\sum_{i=1}^{n} X_iX_i'\right)^{-1} \xrightarrow{p} E\left[X_iX_i'\right]^{-1}$ (assuming $E\left[X_iX_i'\right]$ is nonsingular). Next, we have $\frac{1}{n}\sum_{i=1}^{n} X_iu_i \xrightarrow{p} E\left[X_iu_i\right] = E\left[E\left[X_iu_i|\, X_i\right]\right] = E\left[X_iE\left[u_i|\, X_i\right]\right] = 0$ since

$$E\left[u_i|\, X_i\right] = E\left[\sigma(X_i)\,\varepsilon_i|\, X_i\right] = \sigma(X_i)\,E\left[\varepsilon_i|\, X_i\right] = 0$$

Therefore, by Slutsky's theorem, we have that

$$\hat{\beta}_n^{OLS} \xrightarrow{p} \beta$$

(2) Show that the estimator suggested in (1) is asymptotically normal, and provide its asymptotic covariance matrix.
Solution: Rearranging the above equation, we have

$$\begin{aligned} \hat{\beta}_n^{OLS} - \beta &= \left(\frac{1}{n}\sum_{i=1}^{n} X_iX_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} X_iu_i \\ \sqrt{n}\left(\hat{\beta}_n^{OLS} - \beta\right) &= \underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} X_iX_i'\right)^{-1}}_{(1)} \underbrace{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_iu_i\right)}_{(2)} \end{aligned}$$

By the argument above, $(1) \xrightarrow{p} E\left[X_iX_i'\right]^{-1}$. Next, since the data are i.i.d., and assuming that $Var\left(X_iu_i\right) = E\left[\left(X_iu_i\right)\left(X_iu_i\right)'\right] = E\left[X_iX_i'u_i^2\right] < +\infty$, we have by the central limit theorem

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_iu_i\right) \xrightarrow{d} N\left(0, E\left[X_iX_i'u_i^2\right]\right)$$

Therefore, by Slutsky's theorem we have that

$$\sqrt{n}\left(\hat{\beta}_n^{OLS} - \beta\right) \xrightarrow{d} N\left(0, E\left[X_iX_i'\right]^{-1} E\left[X_iX_i'u_i^2\right] E\left[X_iX_i'\right]^{-1}\right)$$

(3) Discuss whether or not one can estimate γ. If your answer is yes provide the details of the proposed estimator. If the answer is no discuss in detail why not.

Question 4:

Suppose we have a model of the form

$$\begin{aligned} y_i &= \alpha_0 \exp\{x_i'\beta_0\} + \varepsilon_i, \text{ where} \\ E[\varepsilon_i | x_i] &= 0, \end{aligned}$$

for $i = 1, \ldots, n$, where β_0 is a $K \times 1$ vector of parameters and α_0 is a scalar parameter.

(1) Under what condition is α_0 identified?

Solution: We can rewrite the above equation as

$$\begin{aligned} Y_i &= \alpha_0 \exp\{x_i'\beta_0\} + \varepsilon_i \\ &= \exp\{\ln \alpha_0\} \exp\{x_i'\beta_0\} + \varepsilon_i \\ &= \exp\{\ln \alpha_0 + x_i'\beta_0\} + \varepsilon_i \end{aligned}$$

And therefore, as long as X_i

(2) Assume that the condition(s) in (1) is (are) satisfied. Suggest a $K \times 1$ moment function $\varphi_1(y_i, x_i, \alpha, \beta)$ such that

$$E[\varphi_1(y_i, x_i, \alpha_0, \beta_0)] = 0.$$

(3) Suggest an additional $K \times 1$ moment function $\varphi_2(y_i, x_i, \alpha, \beta)$ such that

$$E[\varphi_2(y_i, x_i, \alpha_0, \beta_0)] = 0.$$

(4) Show how to combine the moment functions in (2) and (3) to get the most efficient estimators for $\theta_0 = (\alpha_0, \beta_0')'$, say $\hat{\theta}_n$.

(5) Provide the asymptotic distribution function for the estimator for $\hat{\theta}_n$ obtained in (4).

(6) Suggest a consistent estimator for the asymptotic covariance matrix derived in (5).

Question 5:

Consider the two equation model given by

$$\begin{aligned} y_{1i} &= x_{1i}'\beta_1 + u_{1i}, \\ y_{2i} &= x_{2i}'\beta_2 + u_{2i}, \end{aligned}$$

where x_{1i} and x_{2i} are $K_1 \times 1$ and $K_2 \times 1$ vectors of regressors, respectively, β_1 and β_2 are the corresponding unknown vectors of parameters. Let $u_i = (u_{1i}, u_{2i})'$, and assume that

$$u_i | x_{1i}, x_{2i} \overset{iid}{\sim} D(0, \Sigma_u).$$

(1) Describe how to obtain the most efficient estimates for β_1 and β_2.

(2) Provide a consistent estimator for Σ_u.

(3) Suppose that $Cov(u_{1i}, u_{2i}) = 0$. How would it change the answer in (1)?

(4) Suppose that $x_{1i} = x_{2i}$ for all $i = 1, \ldots, n$ How would it change the answer in (1)?

Question 6:

Consider the two equation model given by

$$\begin{aligned} y_{1i} &= \gamma_1 y_{2i} + x_{1i}'\beta_1 + u_{1i}, \\ y_{2i} &= \gamma_2 y_{1i} + x_{2i}'\beta_2 + u_{2i}, \end{aligned}$$

where x_{1i} and x_{2i} are $K_1 \times 1$ and $K_2 \times 1$ vectors of regressors, respectively, β_1 and β_2 are the corresponding unknown vectors of parameters, and γ_1 and γ_2 are two scalar parameters. Let $u_i = (u_{1i}, u_{2i})'$, and assume that

$$u_i | x_{1i}, x_{2i} \overset{iid}{\sim} D(0, \Sigma_u).$$

(1) Describe in detail the conditions under which the model's parameters are identified.

(2) Assume the conditions in (1) are satisfied. Provide the most efficient estimators for $\theta_1 = (\gamma_1, \beta_1)'$, say $\hat{\theta}_1$.

(3) Derive the asymptotic distribution for $\hat{\theta}_1$.

(4) Suggest a test for the hypothesis $H_0 : \gamma_1 = \gamma_2$ against $H_1 : \gamma_1 \neq \gamma_2$.

Question 7:
Consider the binary discrete choice model given by

$$\Pr\left[y_i = 0 |\, x_i\right] = \frac{\exp\left\{x_i'\gamma\right\}}{1+\exp\left\{x_i'\gamma\right\}},$$

for $i = 1, \ldots, n$.
(1) Provide the MLE for γ, say $\hat{\gamma}_n$.
Solution: Define

$$\Lambda\left(X_i'\gamma\right) = \frac{\exp\left\{X_i'\gamma\right\}}{1+\exp\left\{X_i'\gamma\right\}}$$

Then

$$L_n\left(\gamma; \{X_i\}\right) = \prod_{i=1}^{n} \Lambda\left(X_i'\gamma\right)^{1-Y_i}\left[1-\Lambda\left(X_i'\gamma\right)\right]^{Y_i}$$

Taking logs,

$$\ln L_n\left(\gamma; \{X_i\}\right) = \sum_{i=1}^{n}\left[\left(1-Y_i\right)\ln\Lambda\left(X_i'\gamma\right) + Y_i \ln\left[1-\Lambda\left(X_i'\gamma\right)\right]\right]$$

Taking first order conditions, we have that $\hat{\gamma}_n$ solves

$$(\gamma) : \sum_{i=1}^{n}\frac{1-Y_i}{\Lambda\left(X_i'\hat{\gamma}_n\right)}\lambda\left(X_i'\hat{\gamma}_n\right)X_i - \sum_{i=1}^{n}\frac{Y_i}{1-\Lambda\left(X_i\hat{\gamma}_n\right)}\lambda\left(X_i'\hat{\gamma}_n\right)X_i = 0$$

Where

$$\lambda\left(X_i'\hat{\gamma}_n\right) \equiv \frac{\partial}{\partial X_i'\hat{\gamma}_n}\Lambda\left(X_i'\hat{\gamma}_n\right)$$

Is the pdf of a logistic distribution.

(2) Show that the MLE can be considered as an MM estimator.
Solution:

(3) Compute the asymptotic covariance for $\hat{\gamma}_n$ and provide a consistent estimator for the asymptotic covariance. Justify your answer.
Solution:

Question 8:
Consider the linear model

$$\begin{aligned} y_i &= x_i'\beta + u_i, \text{ where} \\ E\left[u_i | x_i\right] &\neq 0, \end{aligned}$$

for $i = 1, \ldots, n$, where x_i is a $K \times 1$ vector of regressors. Suppose that there exist a vector of random variables z_i such that

$$E\left[u_i | z_i\right] = 0,$$

where z_i is an $M \times 1$ vector, with $M > K$.
(1) Show that a least-squares regression of y_i on x_i will yield an inconsistent estimator for β.
Solution: Since $E\left[u_i | x_i\right] \neq 0$, it follows that $E\left[x_i u_i\right] \neq 0$. Therefore,

$$\begin{aligned} \hat{\beta}_n^{OLS} &= \left(X'X\right)^{-1} X'Y = \left(X'X\right)^{-1} X'\left(X\beta + u\right) \\ &= \beta + \left(X'X\right)^{-1} X'u \\ &= \beta + \left(\frac{1}{n}\sum_{i=1}^{n} x_i x_i'\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} x_i u_i \end{aligned}$$

By the weak law of large numbers, Slutsky's theorem, and the Mann Wald theorem, we have that

$$\hat{\beta}_n^{OLS} \xrightarrow{p} \beta + E\left[x_i x_i'\right]^{-1} E\left[x_i u_i\right] \neq \beta$$

That is, $\hat{\beta}_n^{OLS}$ is inconsistent.

(2) Suggest an instrumental variable estimator for β using the entire vector of instruments z_i.
Solution: Let Π be $k \times \ell$ with $\Pi \xrightarrow{p} \Pi_0$, which is nonstochastic. Then

$$\hat{\beta}_{IV}\left(\Pi\right) = \left(\Pi Z'X\right)^{-1} \Pi Z'Y$$

Is an instrumental variable estimator for β. In particular, if we let $\Pi = X'Z\left(Z'Z\right)^{-1}$, we have that

$$\begin{aligned} \hat{\beta}_{IV}\left(X'Z\left(Z'Z\right)^{-1}\right) &= \left(X'Z\left(Z'Z\right)^{-1} Z'X\right)^{-1} X'Z\left(Z'Z\right)^{-1} Z'Y \\ &\equiv \left(\hat{X}'\hat{X}\right)^{-1} \hat{X}'Y \end{aligned}$$

Is nothing but $\hat{\beta}_{2SLS}$.

(3) Show that the estimator suggested in (2) can be viewed as a GMM estimator.
Solution: Since

$$0 = E\left[u_i | Z_i\right] = E\left[Y_i - X_i'\beta | Z_i\right]$$

Multiplying both sides by Z_i,

$$\begin{aligned} 0 &= Z_i E\left[Y_i - X_i'\beta | Z_i\right] \\ &= E\left[Z_i\left(Y_i - X_i'\beta\right) | Z_i\right] \\ &= E\left[Z_i\left(Y_i - X_i'\beta\right)\right] \end{aligned}$$

Let $\varphi\left(\beta\right) = Z_i\left(Y_i - X_i'\beta\right)$. Then if we let $m_n\left(\beta\right) = \frac{1}{n}\sum_{i=1}^{n}\varphi\left(\beta\right) = \frac{1}{n}\sum_{i=1}^{n} Z_i\left(Y_i - X_i'\beta\right)$, we can define our GMM estimator by letting $V_n^{-1} = \left(Z'Z\right)^{-1}$

$$\hat{\beta}_{GMM} = \arg\min m_n\left(\beta\right)'\left(Z'Z\right)^{-1} m_n\left(\beta\right)$$

Taking first order conditions,

$$(\beta') : 2\frac{\partial m_n\left(\hat{\beta}_{GMM}\right)}{\partial \beta}\left(Z'Z\right)^{-1} m_n\left(\hat{\beta}_{GMM}\right) = 0$$

Or

$$\begin{aligned} 0 &= \frac{1}{n}\sum_{i=1}^{n} X_i Z_i' \left(Z'Z\right)^{-1} \frac{1}{n}\sum_{i=1}^{n} Z_i\left(Y_i - X_i'\hat{\beta}_{GMM}\right) \\ &= \frac{1}{n^2} X'Z\left(Z'Z\right)^{-1} Z'\left(Y - X\hat{\beta}_{GMM}\right) \end{aligned}$$

This gives us

$$X'Z\left(Z'Z\right)^{-1} Z'\left(Y - X\hat{\beta}_{GMM}\right) = 0$$

Rearranging,

$$\begin{aligned} X'Z\left(Z'Z\right)^{-1} Z'Y &= X'Z\left(Z'Z\right)^{-1} Z'X\hat{\beta}_{GMM} \\ X'P_Z Y &= X'P_Z X\hat{\beta}_{GMM} \end{aligned}$$

Or

$$\begin{aligned} \hat{\beta}_{GMM} &= \left(X'P_Z X\right)^{-1} X'P_Z Y \\ &= \left(\hat{X}'\hat{X}\right)^{-1} \hat{X}'Y = \hat{\beta}_{2SLS} \end{aligned}$$

(4) Using the fact established in (3), provide the asymptotic distribution of the estimator for β.
Solution: Recall that

$$\sqrt{n}\left(\hat{\beta}_{GMM} - \beta\right) \xrightarrow{d} N\left(0, \Lambda\right)$$

Where

$$\Lambda = \left(A(\beta) V^{-1}(\beta) A(\beta)'\right)^{-1} A(\beta) V^{-1}(\beta) W(\beta) \left[V^{-1}(\beta)\right]' \left[A(\beta)\right]' \left(A(\beta)\left[V^{-1}(\beta)\right]' A(\beta)'\right)^{-1}$$

$$\begin{aligned} A(\beta) &= \frac{\partial m_0(\beta)}{\partial \beta} = E\left[Z_i X_i'\right] \\ W(\beta) &= E\left[\varphi(\beta)\varphi(\beta)'\right] = E\left[Z_i Z_i'\left(Y_i - X_i'\beta\right)\left(Y_i - X_i'\beta\right)'\right] \\ &= E\left[Z_i Z_i'\varepsilon_i^2\right] \end{aligned}$$

Here, we have $V^{-1}(\beta) = E\left[Z_i Z_i'\right]^{-1}$

(5) Provide a consistent estimator for the asymptotic covariance matrix established in (4). Justify your answer.
Solution: A consistent estimator for $A(\beta)$ is $\frac{1}{n}\sum_{i=1}^{n} Z_i X_i'$. Also, we have that

$$\frac{1}{n}\sum_{i=1}^{n} Z_i Z_i'\hat{\varepsilon}_i^2 \xrightarrow{p} E\left[Z_i Z_i'\varepsilon_i^2\right] = W(\beta)$$

Where $\hat{\varepsilon}_i = \left(Y_i - X_i\hat{\beta}_{GMM}\right)$. Finally, since $\frac{1}{n}\sum_{i=1}^{n} Z_i Z_i' \xrightarrow{p} E\left[Z_i Z_i'\right]$, we have that

$$\begin{aligned} \hat{\Lambda}_n = & \left(\left[\frac{1}{n}\sum_{i=1}^{n} Z_i X_i'\right]\left[\frac{1}{n}\sum_{i=1}^{n} Z_i Z_i'\right]^{-1}\left[\frac{1}{n}\sum_{i=1}^{n} Z_i X_i'\right]'\right)^{-1}\left[\frac{1}{n}\sum_{i=1}^{n} Z_i X_i'\right]\left[\frac{1}{n}\sum_{i=1}^{n} Z_i Z_i'\right]^{-1} \\ & \left[\frac{1}{n}\sum_{i=1}^{n} Z_i Z_i'\hat{\varepsilon}_i^2\right]\left[\frac{1}{n}\sum_{i=1}^{n} Z_i Z_i'\right]^{-1}\left[\frac{1}{n}\sum_{i=1}^{n} Z_i X_i'\right]'\left(\left[\frac{1}{n}\sum_{i=1}^{n} Z_i X_i'\right]\left[\frac{1}{n}\sum_{i=1}^{n} Z_i Z_i'\right]^{-1}\left[\frac{1}{n}\sum_{i=1}^{n} Z_i X_i'\right]'\right)^{-1} \end{aligned}$$

Is a consistent estimator of Λ by Slutsky's theorem, the Mann-Wald theorem, and the laws of large numbers.

Question 9:
Consider the system of equation model given by

$$\begin{aligned} y_{1i} &= \gamma_{12}y_{2i}+\gamma_{13}y_{3i}+x_{1i}'\beta_1+u_{1i}, \\ y_{2i} &= \gamma_{23}y_{3i}+x_{2i}'\beta_2+u_{2i}, \\ y_{3i} &= x_{3i}'\beta_3+u_{3i}, \end{aligned}$$

where x_{1i}, x_{2i}, and x_{3i} are $K_1\times 1$, $K_2\times 1$, and $K_3\times 1$ vectors of exogenous regressors. Let $u_i = (u_{1i},u_{2i},u_{3i})'$ and assume that

$$u_i \overset{iid}{\sim} D(0,\Sigma_u).$$

(1) Assume that $\Sigma_u = diag\left(\sigma_1^2,\sigma_2^2,\sigma_3^2\right)$. Determine which of the above parameters is identified.
Solution: Since $\Sigma_u = diag\left(\sigma_1^2,\sigma_2^2,\sigma_3^2\right)$, we have that $E\left[u_{ji}u_{ki}\right]=0$ for $j\neq k$.
Equation 3: There are no endogeneity problems here, so each parameter can be identified.
Equation 2:

$$\begin{aligned} E\left[y_{3i}u_{2i}\right] &= E\left[\left(x_{3i}'\beta_3+u_{3i}\right)u_{2i}\right] \\ &= E\left[u_{2i}x_{3i}'\right]\beta_3+E\left[u_{3i}u_{2i}\right] \\ &= 0 \end{aligned}$$

That is, y_{3i} is exogenous for equation 2. Therefore, we can identify γ_{23} as well as β_2.
Equation 1:

$$\begin{aligned} E\left[y_{3i}u_{1i}\right] &= E\left[\left(x_{3i}'\beta_3+u_{3i}\right)u_{1i}\right] \\ &= E\left[u_{1i}x_{3i}'\right]\beta_3+E\left[u_{1i}u_{2i}\right] \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} E\left[y_{2i}u_{1i}\right] &= E\left[\left(\gamma_{23}y_{3i}+x_{2i}'\beta_2+u_{2i}\right)u_{1i}\right] \\ &= \gamma_{23}E\left[u_{1i}y_{3i}\right]+E\left[u_{1i}x_{2i}'\right]\beta_2+E\left[u_{2i}u_{1i}\right] \\ &= 0 \end{aligned}$$

Therefore, both y_{2i} and y_{3i} are exogenous for equation 1. Therefore, we can identify γ_{12},γ_{13}, and β_1.

(2) How would your answer to (1) change if the off-diagonal terms of Σ_u are not zero? Explain briefly.
Solution: If the off-diagonal terms are not zero, we need to check the order conditions for each equation. Let Z_i be a vector of unique exogenous regressors. (That is, Z_i contains all the elements of x_{1i}, all the elements of x_{2i} that are not elements of x_{1i}, and all the elements of x_{3i} that are not elements of x_{2i} or x_{1i}.) Let $Z = \begin{bmatrix} Z_1' \\ \dots \\ Z_n' \end{bmatrix}$.
Let ℓ_j be the number of elements
(3) Provide estimators for all the parameters that you determine in (1) to be identified.
Solution: Since there are no exogeneity problems here, we can rewrite

$$\begin{aligned} y_{1i} &= \gamma_{12}y_{2i}+\gamma_{13}y_{3i}+x_{1i}'\beta_1+u_{1i} = W_{1i}'\delta_1+u_{1i} \\ y_{2i} &= \gamma_{23}y_{3i}+x_{2i}'\beta_2+u_{2i} = W_{2i}'\delta_2+u_{2i} \\ y_{3i} &= x_{3i}'\beta_3+u_{3i} = W_{3i}'\delta_3+u_{3i} \end{aligned}$$

And estimate using OLS

$$\begin{aligned} \hat{\delta}_1^{OLS} &= \left(W_1'W_1\right)^{-1}W_1'Y_1 \\ \hat{\delta}_2^{OLS} &= \left(W_2'W_2\right)^{-1}W_2'Y_2 \\ \hat{\delta}_3^{OLS} &= \left(W_3'W_3\right)^{-1}W_3'Y_3 \end{aligned}$$

(4) Provide a consistent estimator for Σ_u.

Solution: Since $\Sigma_u = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix}$, we can estimate

$$\begin{aligned} \hat{\sigma}_1^2 &= \frac{1}{n}\hat{u}_1'\hat{u}_1 \\ \hat{\sigma}_2^2 &= \frac{1}{n}\hat{u}_2'\hat{u}_2 \\ \hat{\sigma}_3^2 &= \frac{1}{n}\hat{u}_3'\hat{u}_3 \end{aligned}$$

Where $\hat{u}_j = Y_{ji} - W_{ji}'\delta_j$.

Question 10:
Let the system of equation model be given by

$$\begin{aligned} y_{1i} &= \gamma_{12} y_{2i} + x_{1i}'\beta_1 + u_{1i}, \\ y_{2i} &= \gamma_{23} y_{3i} + x_{2i}'\beta_2 + u_{2i}, \\ y_{3i} &= \gamma_{31} y_{1i} + x_{3i}'\beta_3 + u_{3i}, \end{aligned}$$

where x_{1i}, x_{2i}, and x_{3i} are $K_1 \times 1, K_2 \times 1$, and $K_3 \times 1$, vectors of exogenous regressors.
(1) Express this system in the concise form

$$y_i'\Gamma = x_i'B + u_i', \tag{1}$$

where y_i is a $J \times 1$ vector, Γ is a $J \times J$ matrix of coefficients, and B is a $K \times J$ matrix of coefficients.
Solution: Rearranging, we get

$$\begin{aligned} y_{1i} - \gamma_{12} y_{2i} &= x_{1i}'\beta_1 + u_{1i}, \\ y_{2i} - \gamma_{23} y_{3i} &= x_{2i}'\beta_2 + u_{2i}, \\ -\gamma_{31} y_{1i} + y_{3i} &= x_{3i}'\beta_3 + u_{3i}, \end{aligned}$$

Or, if we assume that x_{1i}, x_{2i}, and x_{3i} have no elements in common,

$$\begin{bmatrix} y_{1i} & y_{2i} & y_{3i} \end{bmatrix} \begin{bmatrix} 1 & -\gamma_{12} & 0 \\ 0 & 1 & -\gamma_{23} \\ -\gamma_{31} & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_{1i}' & x_{2i}' & x_{3i}' \end{bmatrix} \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{bmatrix} + \begin{bmatrix} u_{1i} & u_{2i} & u_{3i} \end{bmatrix}$$

(2) Verify whether or not the order condition for identification holds for the above system of equations.
Solution: Stacking the above equations, we have

$$Y\Gamma = XB + E$$

Rearranging,

$$\begin{aligned} Y &= Y - Y\Gamma + XB + E \\ &= \begin{bmatrix} Y & X \end{bmatrix} \begin{bmatrix} I - \Gamma \\ B \end{bmatrix} + E \\ &= \begin{bmatrix} Y & X \end{bmatrix} A + E \end{aligned}$$

Where

$$A = \begin{bmatrix} I - \Gamma \\ B \end{bmatrix} = \begin{bmatrix} 0 & \gamma_{12} & 0 \\ 0 & 0 & \gamma_{23} \\ \gamma_{31} & 0 & 0 \\ \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{bmatrix}$$

Let R_j be the number of restrictions in the jth column of A. Here, we have

$$\begin{aligned} R_1 &= 2 + K_2 + K_3 \\ R_2 &= 2 + K_1 + K_3 \\ R_3 &= 2 + K_1 + K_2 \end{aligned}$$

If $R_1 \geq 3$, we have identification of the first equation. That is, if

$$\begin{aligned} 2 + K_2 + K_3 &\geq 3 \\ K_2 + K_3 &\geq 1 \end{aligned}$$

Similarly, if $R_2 \geq 3$, we have identification of the second equation. This occurs if

$$\begin{aligned} 2 + K_1 + K_3 &\geq 3 \\ K_1 + K_3 &\geq 1 \end{aligned}$$

Finally, if $R_3 \geq 3$, we have identification of the third equation.

$$\begin{aligned} 2 + K_1 + K_2 &\geq 3 \\ K_1 + K_2 &\geq 1 \end{aligned}$$

(3) Provide the reduced-form of the model and explain how to estimate efficienty the reduced form parameters.

Solution: If we postmultiply both sides of (1) by Γ^{-1}, we have

$$\begin{aligned} y_i \Gamma \Gamma^{-1} &= x_i' B \Gamma^{-1} + u_i' \Gamma^{-1} \\ y_i &= x_i' \Pi + u_i' \Gamma^{-1} \\ &= x_i' \Pi + \varepsilon_i' \end{aligned}$$

Here, since $E[x_i \varepsilon_i'] = 0$, we have that OLS will be consistent. Next, we note that

$$\begin{aligned} V(\varepsilon_i | X_i) &= V\left(\left(\Gamma^{-1}\right)' u_i \middle| X_i \right) = \left(\Gamma^{-1}\right)' V(u_i | X_i) \Gamma^{-1} \\ \Omega &= \left(\Gamma^{-1}\right)' \Sigma \Gamma^{-1} \end{aligned}$$

Then if we estimate using GLS,

$$\hat{\Pi}_n^{GLS} = \left(X' \Omega^{-1} X\right)^{-1} X' \Omega^{-1} Y$$

we will have an efficient estimator for the reduced form parameters.

(4) Suggest a test for the null hypothesis $H_0 : \gamma_{12} = \gamma_{23} = \gamma_{31}$, against H_1 : Not H_0.

Solution: Let

$$\begin{aligned} y_{1i} &= \gamma_{12} y_{2i} + x_{1i}' \beta_1 + u_{1i} \equiv W_{1i}' \delta_1 + u_{1i} \\ y_{2i} &= \gamma_{23} y_{3i} + x_{2i}' \beta_2 + u_{2i} \equiv W_{2i}' \delta_2 + u_{2i} \\ y_{3i} &= \gamma_{31} y_{1i} + x_{3i}' \beta_3 + u_{3i} \equiv W_{3i}' \delta_3 + u_{3i} \end{aligned}$$

Stacking this, we have

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} W_{1i}' & 0 & 0 \\ 0 & W_{2i}' & 0 \\ 0 & 0 & W_{3i}' \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Where

$$\delta_1 = \begin{bmatrix} \gamma_{12} \\ \beta_1 \end{bmatrix}, \ \delta_2 = \begin{bmatrix} \gamma_{23} \\ \beta_2 \end{bmatrix}, \ \delta_3 = \begin{bmatrix} \gamma_{31} \\ \beta_3 \end{bmatrix}$$

We can then estimate this using 2SLS, since we have assumed that it is identified. That is, we can estimate

$$\hat{\delta}_n^{2SLS} = \begin{bmatrix} \hat{\delta}_1^{2SLS} \\ \hat{\delta}_2^{2SLS} \\ \hat{\delta}_3^{2SLS} \end{bmatrix} = \left(\hat{W}' \hat{W}\right)^{-1} \hat{W}' Y$$

Where

$$\hat{W} = \left(I \otimes X \left(X'X\right)^{-1} X'\right) W$$

The hypothesis we are asked to test is

$$
\begin{aligned}
H_0 &: \Gamma B = 0 \\
H_0 &: \begin{bmatrix} 1 & 0_{K_1\times 1} & 0_{1\times 1} & 0_{K_2\times 1} & 0_{1\times 1} & 0_{K_3\times 1} \\ 0_{1\times 1} & 0_{K_1\times 1} & 1 & 0_{K_2\times 1} & 0_{1\times 1} & 0_{K_3\times 1} \\ 0_{1\times 1} & 0_{K_1\times 1} & 0_{1\times 1} & 0_{K_2\times 1} & 1 & 0_{K_3\times 1} \end{bmatrix} \begin{bmatrix} \gamma_{12} \\ \beta_1 \\ \gamma_{23} \\ \beta_2 \\ \gamma_{31} \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

Let A be the asymptotic variance-covariance matrix for $\hat{\delta}_n^{2SLS}$. Then we use the test statistic

$$
W_0 = n\left(\Gamma\hat{\delta}_n^{2SLS}\right)'\left[\Gamma A \Gamma'\right]\left(\Gamma\hat{\delta}_n^{2SLS}\right) \xrightarrow{d} \chi^2\left(3\right)
$$