OH - Thursday, 10-12 pm

Fall 2003 comp, Q 2 Buchinsky

$$Pr[Y_i = 0 \mid X_i] = \frac{\exp\{X_i'\gamma\}}{1+\exp\{X_i'\gamma\}}$$

d) $$\min \sum_{i=1}^{n} \frac{(Y_i - f_{Y_i}(Y_i \mid X_i))^2}{f_{Y_i}(Y_i \mid X)(1 - f_{Y_i}(Y_i \mid X))}$$. Is the resulting estimator consistent?

Digression: LPM

assume $Pr[Y_i = 0 \mid X_i] = X_i'\tilde{\gamma}$

$$\Rightarrow E[Y_i \mid X_i] = 0 \cdot Pr[Y_i = 0 \mid X_i] + 1 \cdot Pr[Y_i = 1 \mid X_i]$$
$$= 1 - X_i'\tilde{\gamma}$$
$$= X_i'\gamma$$ assuming X contains a constant

$$Y_i = E[Y_i \mid X_i] + (Y_i - E[Y_i \mid X_i])$$
$$= X_i'\gamma + \varepsilon_i$$

$$E[\varepsilon_i \mid X_i] = E[Y_i - E[Y_i \mid X_i] \mid X_i]$$
$$= E[Y_i \mid X_i] - E[Y_i \mid X_i] = 0$$, so OLS will be consistent

$$\Rightarrow \hat{\gamma}_{OLS} = \arg\min \sum_{i=1}^{n} (Y_i - X_i'\gamma)^2$$

$$\varepsilon_i = Y_i - X_i'\gamma$$
$$= \begin{cases} 1 - X_i'\gamma & Pr[Y_i = 1 \mid X_i] \\ -X_i'\gamma & Pr[Y_i = 0 \mid X_i] \end{cases}$$

$$V(\varepsilon_i | X_i) = \Pr[\varepsilon_i = 1 | X_i](1 - X_i'\gamma)^2 + \Pr[\varepsilon_i = 0](X_i'\gamma)^2$$
$$= (1 - X_i'\gamma) X_i'\gamma$$

$$\Rightarrow \hat{\gamma}_{WLS} = \operatorname{argmin} \sum_{i=1}^{n} \frac{(Y_i - \Pr[Y_i = 1 | X_i])^2}{\Pr[Y_i = 1 | X_i](1 - \Pr[Y_i = 1 | X_i])}$$

End of digression

$$\Pr[Y_i = 0 | X_i] = \frac{\exp\{X_i'\gamma\}}{1 + \exp\{X_i'\gamma\}}$$

$$E[Y_i | X_i] = 0 \cdot \Pr[Y_i = 0 | X_i] + 1 \cdot \Pr[Y_i = 1 | X_i]$$
$$= \frac{1}{1 + \exp\{X_i'\gamma\}}$$
$$\equiv g(X_i, \gamma) \equiv g_i$$

$$Y_i = g_i + \underbrace{(Y_i - g_i)}_{u_i}$$
$$= g_i + u_i$$

where $E[u_i | X_i] = E[Y_i - g_i | X_i] = E[Y_i | X_i] - g_i$
$$= g_i - g_i = 0$$

$$\min \sum_{i=1}^{n} (Y_i - \Pr[Y_i | X_i])^2$$

where $\Pr[Y_i | X_i] = [\Pr[Y_i = 0 | X_i]]^{1 - Y_i} [\Pr[Y_i = 1 | X_i]]^{Y_i}$

Review question 6:

$$Y_{1i} = \gamma_1 Y_{2i} + X_{1i}'\beta_1 + u_{1i} \qquad \begin{bmatrix} u_{1i} \\ u_{2i} \end{bmatrix} \Big| X_{1i}, X_{2i} \sim D(0, \Sigma_u)$$

$$Y_{2i} = \gamma_2 Y_{1i} + X_{2i}'\beta_2 + u_{2i}$$

assume $X_{1i} \neq X_{2i}$ but $\dim(X_{1i}) = \dim(X_{2i}) = 1$

How do you identify the parameters?

(2) 3SLS

$$Y_j = X_j \beta_j + \varepsilon_j \qquad j = 1, 2$$

where $Y_1 = \begin{bmatrix} Y_{11} \\ \vdots \\ Y_{1n} \end{bmatrix}$, $X_1 = \begin{bmatrix} Y_{21} & X_{11}' \\ \vdots & \vdots \\ Y_{2n} & X_{1n}' \end{bmatrix}_{n \times (k_1+1)}$, $\varepsilon_1 = \begin{bmatrix} \varepsilon_{11} \\ \vdots \\ \varepsilon_{1n} \end{bmatrix}$

$Y_2 = \begin{bmatrix} Y_{21} \\ \vdots \\ Y_{2n} \end{bmatrix}$, $X_2 = \begin{bmatrix} Y_{11} & X_{21}' \\ \vdots & \vdots \\ Y_{1n} & X_{2n}' \end{bmatrix}$, $\varepsilon_2 = \begin{bmatrix} \varepsilon_{21} \\ \vdots \\ \varepsilon_{2n} \end{bmatrix}$

Instruments: $Z = \begin{bmatrix} X_{11}' & X_{21}' \\ \vdots & \vdots \\ X_{1n}' & X_{2n}' \end{bmatrix}_{n \times (k_1+k_2)}$

Let $Y = X\beta + \varepsilon \iff \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$

assume $E[\varepsilon_{ji} | Z_i] = 0$

$$E[\varepsilon_{ji}\varepsilon_{ki'}] = \begin{cases} \sigma_{jk} & i = i' \\ 0 & \text{else} \end{cases}$$

2SLS

[1] Regress X on Z, get $\hat{X}$

$$\hat{X} = (I_2 \otimes Z(Z'Z)^{-1}Z')X$$

$$= \begin{bmatrix} Z(Z'Z)^{-1}Z'X_1 & 0 \\ 0 & Z(Z'Z)^{-1}Z'X_2 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{Y}_{11} \quad X_{11}' & 0 \\ \hat{Y}_{1n} \quad X_{1n}' & \\ & \hat{Y}_{21} \quad X_{21}' \\ 0 & \hat{Y}_{2n} \quad X_{2n}' \end{bmatrix}$$

[2] Regress Y on $\hat{X}$

$$\hat{\beta}_{2SLS} = (\hat{X}'\hat{X})^{-1}\hat{X}'Y$$

or $$\begin{bmatrix} \hat{\beta}_{1,2SLS} \\ \hat{\beta}_{2,2SLS} \end{bmatrix} = \begin{bmatrix} (X_1'P_Z X_1)^{-1} X_1' P_Z Y_1 \\ (X_2'P_Z X_2)^{-1} X_2' P_Z Y_2 \end{bmatrix}$$

But this need not be consistant. Consider:

$$\hat{\beta}_{G2SLS} = (\hat{X}'V_0^{-1}\hat{X})^{-1}\hat{X}'V_0^{-1}Y$$

where $V_0 \equiv \Sigma_u \otimes I_n$. But we do not know Σ_u.

$$\hat{\beta}_{3SLS} \equiv \hat{\beta}_{FG2SLS} = (\hat{X}'\hat{V}^{-1}\hat{X})^{-1}\hat{X}'\hat{V}^{-1}Y$$

where $\hat{V} = \hat{\Sigma}_u \otimes I_n$

(3) $\sqrt{n}(\hat{\beta}_{3SLS}-\beta) = \frac{1}{n}(\hat{X}'\hat{V}^{-1}\hat{X})^{-1}\frac{1}{\sqrt{n}}\hat{X}'\hat{V}^{-1}u \xrightarrow{d} N(0,\Omega)$

by CLT, where

$\Omega = (\hat{X}'\hat{V}^{-1}\hat{X})^{-1}$

(4) $H_0: \gamma_1 = \gamma_2 \iff H_0: \gamma_1 - \gamma_2 = 0$

$H_1: \gamma_1 - \gamma_2 \neq 0$

$\iff H_0: \Gamma\beta = 0 \iff \begin{array}{c} \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \end{bmatrix} \\ \text{1st pos.} \qquad\qquad k_1+2 \text{ position} \end{array} \begin{bmatrix} \gamma_1 \\ \beta_1 \\ \gamma_2 \\ \beta_2 \end{bmatrix} = 0$

$$t_0 = \frac{\Gamma\hat{\beta}_{3SLS}}{\sqrt{\Gamma\Omega\Gamma'}} \xrightarrow{d} N(0,1)$$