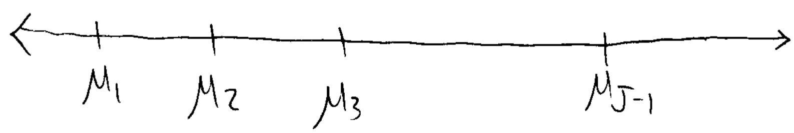
Review session Wednesday, 3:30pm

Ordered response model:

$Y_i^* = X_i'\beta + \varepsilon_i$ Latent variable

- we are trying to estimate β.

There is a sequence $-\infty < \mu_1 < \mu_2 < \ldots < \mu_{J-1} < +\infty$ of constants which partition the real line.



Choose j $(j = 0, \ldots, J-1)$ if $\mu_j < Y_i^* < \mu_{j+1}$

ie $Y_i = j$ $\quad \mu_j < Y_i^* < \mu_{j+1}$

What is $\Pr[Y_i = j \mid X_i]$?

$$\Pr[Y_i = j \mid X_i] = \Pr[\mu_j < Y_i^* < \mu_{j+1} \mid X_i]$$
$$= \Pr[Y_i^* < \mu_{j+1} \mid X_i] - \Pr[Y_i^* < \mu_j \mid X_i]$$
$$= \Pr[\varepsilon_i < \mu_{j+1} - X_i'\beta \mid X_i] - \Pr[\varepsilon_i < \mu_j - X_i'\beta \mid X_i]$$

Suppose $\varepsilon_i \mid X_i \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$

$$\Rightarrow \Pr[Y_i = j \mid X_i] = \Phi\left(\frac{\mu_{j+1} - X_i'\beta}{\sigma_\varepsilon}\right) - \Phi\left(\frac{\mu_j - X_i'\beta}{\sigma_\varepsilon}\right)$$

Detour:

Binary $\varepsilon_i \mid X_i \sim N(0, \sigma_\varepsilon^2)$

$\Rightarrow \Pr[Y_i = 1 \mid X_i] = \Phi\left(\frac{X_i'\beta}{\sigma_\varepsilon}\right)$ cannot estimate β and σ_ε separately

Consider the following model: $\varepsilon_i | X_i \sim N(0, \sigma_\varepsilon^{2*})$, $\sigma_\varepsilon^{2*} = 2\sigma_\varepsilon^2$
and $\beta^* = \sqrt{2}\beta$.

Then $\Pr[Y_i = 1 | X_i] = \Phi\left(X_i' \frac{\sqrt{2}\beta}{\sqrt{2}\sigma_\varepsilon}\right) = \Phi\left(X_i' \frac{\beta}{\sigma_\varepsilon}\right)$

- observationally equivalent
  - cannot identify β^*, σ_ε^{2*}

Define $z_{ji} = \begin{cases} 1 & \text{if } Y_i = j \\ 0 & \text{else} \end{cases}$

$\Rightarrow \sum_{j=0}^{J-1} z_{ji} = 1$

This gives us

$$\ln L_n(\beta, \mu) = \sum_{i=1}^{n} \left\{ \sum_{j=0}^{J-1} z_{ji} \ln \left[\Phi\left(\frac{\mu_{j+1} - X_i'\beta}{\sigma_\varepsilon} \right) - \Phi\left(\frac{\mu_j - X_i'\beta}{\sigma_\varepsilon} \right) \right] \right\}$$

Random utility model (unordered response model).

Let $Y_{ji}^* = X_{ji}'\beta_j + \varepsilon_{ji}$ be the utility derived from choice j
$j = 1, \ldots, J$

In the ordered model, regardless of the number of choices, there was only one ε_i. Here there are J.

Define $Y_{ji} = \begin{cases} 1 & \text{if } j \text{ is taken} \\ 0 & \text{else} \end{cases}$

Clearly, $\sum_{j=1}^{J} Y_{ji} = 1 \quad \forall i$

$$Pr[Y_{ji}=1 \mid X_i] = Pr[Y_{ji}^* > Y_{ki}^*, j \neq k \mid X_i]$$
$$= Pr[X_{ji}'\beta_j + \varepsilon_{ji} > X_{ki}'\beta_k + \varepsilon_{ki}, j \neq k \mid X_i]$$

assume $\varepsilon_i \mid X_i = \begin{bmatrix} \varepsilon_{1i} \\ \vdots \\ \varepsilon_{Ji} \end{bmatrix} \mid X_i$

$\Rightarrow Pr[Y_{ji}=1 \mid X_i] = \int \cdots \int f_{\varepsilon_i \mid X_i}(\cdot)\, d\varepsilon_{1i} \cdots d\varepsilon_{Ji}$

This is a huge integral

We may compute this by crude frequency simulation

Example with $J=2$

$Y_{1i}^* = X_{1i}'\beta_1 + \varepsilon_{1i}$

$Y_{2i}^* = X_{2i}'\beta_2 + \varepsilon_{2i}$

$\Rightarrow Y_{1i} = 1$ iff $Y_{1i}^* \geq Y_{2i}^*$

$Y_{2i} = 1$ iff $Y_{2i}^* > Y_{1i}^*$

Define $w_i^* = Y_{1i}^* - Y_{2i}^* = X_{1i}'\beta_1 + \varepsilon_{1i} - X_{2i}'\beta_2 - \varepsilon_{2i}$

Then $Y_{1i}=1$ iff $w_i^* \geq 0$ iff $\overbrace{\varepsilon_{1i} - \varepsilon_{2i}}^{u_i} \geq X_{2i}'\beta_2 - X_{1i}'\beta_1$

iff $\overbrace{\varepsilon_{1i} - \varepsilon_{2i}}^{u_i} \geq Z_i'\gamma$ where Z_i contains all the variables that are in X_{1i} and X_{2i} (w/out duplication)

Define $w_i = \begin{cases} 1 & \text{if } w_i^* \geq 0 \\ 0 & \text{else} \end{cases}$

$$\Rightarrow \Pr[Y_{1i}=1 \mid X_i] = \Pr[W_i = 1 \mid X_i] = \Pr[W_i^* \geq 0 \mid X_i]$$
$$= \Pr[u_i \geq z_i'\gamma \mid X_i]$$

If $\varepsilon_i \mid X_i \overset{iid}{\sim} N(0, \Omega)$, we have a multinomial probit model.

If $\varepsilon_i \mid X_i \overset{iid}{\sim}$ Extreme value (Type I), then we have a multinomial logit model.

- If $\varepsilon_{ji} \mid X_i$, $\varepsilon_{ki} \mid X_i$ have extreme value distribution: $(\varepsilon_{ji} - \varepsilon_{ki}) \mid X_i \overset{iid}{\sim} \Lambda$
- $\Pr[Y_{ji}=1 \mid X_i] = \dfrac{\exp\{X_{ji}'\beta_j\}}{\sum_{k=1}^{J} \exp\{X_{ki}'\beta_k\}}$

This has the property of independence of irrelevant alternatives.

$$\frac{\Pr[Y_{ji}=1 \mid X_i]}{\Pr[Y_{ki}=1 \mid X_i]} = \frac{\exp\{X_{ji}'\beta_j\}}{\exp\{X_{ki}'\beta_k\}} \neq f(X_{\ell i})$$