$\ell(\theta) = \frac{1}{n} \ln L_n(\theta) + \lambda_n' r(\theta)$

(θ): $\underbrace{\frac{1}{n} \frac{\partial}{\partial\theta} \ln L_n(\hat{\theta}_n^R)}_{(a)} + R(\hat{\theta}_n^R)\hat{\lambda}_n = 0$

(λ'): $r(\hat{\theta}^r) = 0$ (1)

Expanding (1) around θ_0:

$0 = r(\hat{\theta}_n^r) = \underbrace{r(\theta_0)}_{=0 \text{ under null}} + R(\tilde{\theta})(\hat{\theta}_n^R - \theta_0)$; $\tilde{\theta} \in |\hat{\theta}_n^R, \theta_0|$

$\Rightarrow R(\tilde{\theta})\sqrt{n}(\hat{\theta}_n^R - \theta_0) = 0$ (2)

Expanding (a) around θ_0:

$\frac{1}{n} \frac{\partial}{\partial\theta} \ln L_n(\hat{\theta}_n^R) = \frac{1}{n} \frac{\partial}{\partial\theta} \ln L_n(\theta_0) + \frac{1}{n} \frac{\partial^2}{\partial\theta\partial\theta'} \ln L_n(\bar{\theta})(\hat{\theta}_n^R - \theta_0)$, $\bar{\theta} \in |\hat{\theta}_n^R, \theta_0|$

$\Rightarrow 0 = \sqrt{n}\frac{1}{n}\frac{\partial}{\partial\theta} \ln L_n(\theta_0) + \underbrace{\frac{1}{n}\frac{\partial^2}{\partial\theta\partial\theta'} \ln L_n(\bar{\theta})}_{\to -I(\theta_0)} \sqrt{n}(\hat{\theta}_n^R - \theta_0) + \underbrace{R(\hat{\theta}_n^R)}_{\to R(\theta_0)} \sqrt{n}\hat{\lambda}_n$

$\Rightarrow 0 \text{ "=" } \sqrt{n}\frac{1}{n}\frac{\partial}{\partial\theta} \ln L_n(\theta_0) - I(\theta_0)\sqrt{n}(\hat{\theta}_n^R - \theta_0) + R(\theta_0)\sqrt{n}\hat{\lambda}_n$

$\Rightarrow 0 = \overbrace{R(\theta_0)' I^{-1}(\theta_0) \underbrace{\frac{1}{\sqrt{n}}\frac{\partial}{\partial\theta} \ln L_n(\theta_0)}_{\xrightarrow{d} N(0, I(\theta_0))}}^{(A)} - \overbrace{R(\theta_0)'\sqrt{n}(\hat{\theta}_n^R - \theta_0)}^{(B)}$

$+ R(\theta_0)' I^{-1}(\theta_0) R(\theta_0) \sqrt{n}\hat{\lambda}_n$

$(A) \xrightarrow{d} N(0, R(\theta_0)' I^{-1}(\theta_0) R(\theta_0))$

By (2), $(B) \to 0$

$\Rightarrow \sqrt{n}\hat{\lambda}_n \text{ "=" } -[R(\theta_0)' I^{-1}(\theta_0) R(\theta_0)]^{-1} A$

$\xrightarrow{d} N(0, [R(\theta_0)' I^{-1}(\theta_0) R(\theta_0)]^{-1})$

This gives us

$$\sqrt{n}\,[R(\theta_0)' I^{-1}(\theta_0) R(\theta_0)]^{1/2}\, \hat{\lambda}_n \xrightarrow{d} N(0, I_q)$$

$$\Rightarrow n\, \hat{\lambda}_n' [R(\theta_0)' I^{-1}(\theta_0) R(\theta_0)]\, \hat{\lambda}_n \xrightarrow{d} \chi^2(q)$$

Define $LM_n \equiv n\, \hat{\lambda}_n' [R(\theta_0)' I^{-1}(\theta_0) R(\theta_0)]\, \hat{\lambda}_n$

From (11.7), $\frac{1}{n}\frac{\partial}{\partial\theta} \ln L_n(\hat{\theta}_n^R) + R(\hat{\theta}_n^R)\hat{\lambda}_n = 0$

$$\Rightarrow \sqrt{n} R(\hat{\theta}_n^R)\hat{\lambda}_n = -\sqrt{n}\,\frac{1}{n}\frac{\partial}{\partial\theta} \ln L_n(\hat{\theta}_n^R)$$

$$\Rightarrow LM_n = \sqrt{n}\,[R(\hat{\theta}_n^R)\hat{\lambda}_n]' I^{-1}(\theta_0) \sqrt{n}\,[R(\hat{\theta}_n^R)\hat{\lambda}_n]$$

$$= \sqrt{n}\,\frac{1}{n}\left[\frac{\partial}{\partial\theta} \ln L_n(\hat{\theta}_n^R)\right]' I^{-1}(\theta_0)\sqrt{n}\left[\frac{1}{n}\frac{\partial}{\partial\theta} \ln L_n(\hat{\theta}_n^R)\right]$$

$$= \frac{1}{n}\left[\frac{\partial}{\partial\theta} \ln L_n(\hat{\theta}_n^R)\right]' I^{-1}(\theta_0)\left[\frac{\partial}{\partial\theta} \ln L_n(\hat{\theta}_n^R)\right] \xrightarrow{d} \chi^2(q)$$

$$\widehat{LM}_n = \frac{1}{n}\left[\frac{\partial}{\partial\theta} \ln L_n(\hat{\theta}_n^R)\right]' I^{-1}(\hat{\theta}_n^R)\left[\frac{\partial}{\partial\theta} \ln L_n(\hat{\theta}_n^R)\right] \xrightarrow{d} \chi^2(q)$$

Likelihood Ratio test

$$LR_n = -2\ln\left[\frac{L_n(\hat{\theta}_n^R)}{L_n(\hat{\theta}_n^{UR})}\right] = 2\left[\ln L_n(\hat{\theta}_n^{UR}) - \ln L_n(\hat{\theta}_n^R)\right]$$

By a 2nd order Taylor expansion,

$$\ln L_n(\hat{\theta}_n^R) = \ln L_n(\hat{\theta}_n^{UR}) + \frac{\partial}{\partial\theta}\ln L_n(\hat{\theta}_n^{UR})(\hat{\theta}_n^{UR} - \hat{\theta}_n^R) + \frac{1}{2}(\hat{\theta}_n^{UR} - \hat{\theta}_n^R)' \frac{\partial^2}{\partial\theta\partial\theta'}\ln L_n(\hat{\theta})\,(\hat{\theta}_n^{UR} - \hat{\theta}_n^R)$$

$$\hat{\theta} \in (\hat{\theta}_n^{UR}, \hat{\theta}_n^R)$$