$\frac{1}{n}\sum \varphi(\hat{\theta})\varphi(\hat{\theta})' \xrightarrow{p} W(\theta_0)$. This will be shown in the second year.

asymptotic tests

LM, LR, and Wald are asymptotically equivalent but they vary widely in the finite sample case.

They are also sensitive to the form of the null hypothesis

$H_0: \theta_1\theta_2 - 1 = 0$, $H_0: \theta_1 - \frac{1}{\theta_2} = 0$ are mathematically equivalent but will yield different values.

assume $f(y, x; \theta_0) = f(y; x; \theta_0)\, g(x)$

In the sample, we might use instead $h(y, x; \theta) = f(y, x, \theta)\, h(x)$ due to oversampling, etc.

Consider : $H_0: r(\theta_0) = 0$ This is a <u>very</u> general null hypothesis.
$H_1: r(\theta_0) \neq 0$

Wald: based on $\hat{\theta}_n^{UR}$
LM: based on $\hat{\theta}_n^{R}$
LR: based on both $\hat{\theta}_n^{UR}, \hat{\theta}_n^{R}$

Recall: $\hat{\theta}_n^{UR} \xrightarrow{p} \theta_0$ and $\sqrt{n}(\hat{\theta}_n^{UR} - \theta_0) \xrightarrow{d} N(0, \Lambda_0)$

assume $\hat{\Lambda}_n \xrightarrow{p} \Lambda_0$

Suppose we want to test $H_0: \underset{q\times 1}{r(\theta_0)} = 0$. It should be the case that $r(\hat{\theta}_n)$ is close to 0 if H_0 is true. What is the asymptotic distribution of $r(\hat{\theta}_n^{UR})$?

By MVT,

$$r(\hat{\theta}_n^{UR}) = r(\theta_0) + \frac{\partial r}{\partial \theta'}\Big|_{\tilde{\theta}} (\hat{\theta}_n^{UR} - \theta_0) \quad \text{where } \tilde{\theta} \in (\hat{\theta}_n^{UR}, \theta_0)$$

$$\Rightarrow \sqrt{n}\,(r(\hat{\theta}_n^{UR}) - r(\theta_0)) = \frac{\partial r}{\partial \theta'}\Big|_{\tilde{\theta}} \sqrt{n}\,(\hat{\theta}_n^{UR} - \theta_0)$$

assume $\underbrace{\left.\frac{\partial r}{\partial \theta'}\right|_{\theta_0}}_{q \times k}$ is s.t. $\operatorname{rank}\left(\left.\frac{\partial r}{\partial \theta'}\right|_{\theta_0}\right) = q$

ie the restrictions are linearly independent.

Define $R(\theta_0)' \equiv \left.\frac{\partial r}{\partial \theta'}\right|_{\theta_0} \equiv \frac{\partial r(\theta_0)}{\partial \theta'}$

Thus, since $\sqrt{n}(\hat{\theta}_n^{UR} - \theta_0) \xrightarrow{d} N(0, \Delta_0)$, we have that

$$\sqrt{n}\,(\underbrace{r(\hat{\theta}_n^{UR})}_{q \times 1} - r(\theta_0)) = R(\bar{\theta})'\sqrt{n}\,(\hat{\theta}_n^{UR} - \theta_0)$$
$$\xrightarrow{d} N(0, \underbrace{R(\theta_0)'}_{q \times k} \underbrace{\Delta_0}_{k \times k} \underbrace{R(\theta_0)}_{k \times q})$$

since $\bar{\theta} \xrightarrow{p} \theta_0$.

(We have just derived the delta method) Under the null, since $r(\theta_0) = 0$, $\sqrt{n}\, r(\hat{\theta}_n^{UR}) \xrightarrow{d} N(0, R(\theta_0)'\Delta_0 R(\theta_0))$

Recall: If $X \sim N(0, I_m)$, $X'X \sim \chi^2(m)$

Recall: If Z is pd and symmetric, $\exists\, P$ s.t. $PP' = Z$

$$\Rightarrow Z_n \equiv \sqrt{n}\,[R(\theta_0)'\Delta_0 R(\theta_0)]^{-1/2}\, r(\hat{\theta}_n^{UR}) \xrightarrow{d} N(0, I_q)$$

$$\Rightarrow \text{Wald}_n \equiv r(\hat{\theta}_n^{UR})'\,[R(\theta_0)'\Delta_0 R(\theta_0)]^{-1}\, r(\hat{\theta}_n^{UR}) \xrightarrow{d} \chi^2(q)$$

Note: If H_0 is not true, $\text{Wald}_n \not\xrightarrow{d} \chi^2(q)$

But we cannot use this statistic, since we do not know $R(\theta_0)'\Delta_0 R(\theta_0)$. We have an estimator $\hat{\Delta}_n \xrightarrow{p} \Delta_0$. also, if R is continuous at θ_0, $R(\hat{\theta}_n^{UR}) \xrightarrow{p} R(\theta_0)$. Thus, $R(\hat{\theta}_n^{UR})'\hat{\Delta}_n R(\hat{\theta}_n^{UR}) \xrightarrow{p} R(\theta_0)'\Delta_0 R(\theta_0)$

Restricted estimator

$$\hat{\theta}_n^R = \underset{\theta : r(\theta)=0}{\operatorname{argmax}} \ \frac{1}{n} \ln L_n(\theta) \qquad \text{for MLE}$$

$$\hat{\theta}_n^R = \underset{\theta : r(\theta)=0}{\operatorname{argmin}} \ m_n(\theta)' V_n^{-1} m_n(\theta) \qquad \text{For GMM}$$

Let $\mathcal{L}(\theta) = \frac{1}{n} \ln L_n(\theta) + \lambda' r(\theta)$

The FOCs are

$$\frac{1}{n} \frac{\partial}{\partial \theta} \ln L_n(\hat{\theta}_n^R) + R(\hat{\theta}_n^R)\hat{\lambda}_n = 0 \quad \text{and}$$

$$r(\hat{\theta}_n^R) = 0$$

By the mean value theorem, for some $\tilde{\theta} \in |\hat{\theta}_n^R, \theta_0|$

$$0 = \frac{1}{n} \frac{\partial}{\partial \theta} \ln L_n(\theta_0) + \frac{1}{n} \frac{\partial^2}{\partial \theta \partial \theta'} \ln L_n(\tilde{\theta})(\hat{\theta}_n^R - \theta_0) + \underset{k \times q}{R(\hat{\theta}_n^R)} \underset{q \times 1}{\hat{\lambda}_n}$$

$$\Rightarrow 0 = \sqrt{n}\left[\frac{1}{n} \frac{\partial}{\partial \theta} \ln L_n(\theta_0)\right] + \left[\frac{1}{n} \frac{\partial^2}{\partial \theta \partial \theta'} \ln L_n(\tilde{\theta}) \sqrt{n}(\hat{\theta}_n^R - \theta_0)\right] + R(\hat{\theta}_n^R)\sqrt{n}\hat{\lambda}_n$$

also, $0 = \underbrace{r(\theta_0)}_{=0} + \left.\frac{\partial r}{\partial \theta'}\right|_{\hat{\hat{\theta}}} (\hat{\theta}_n^R - \theta_0)$

$$\Rightarrow 0 = R(\hat{\hat{\theta}})' \sqrt{n}(\hat{\theta}_n^R - \theta_0), \quad \hat{\hat{\theta}} \in |\hat{\theta}_n^R, \theta_0|$$

Note that $\frac{1}{n} \frac{\partial^2}{\partial \theta \partial \theta'} \ln L_n(\tilde{\theta}) \xrightarrow{p} -I(\theta_0)^{-1}$

$$R(\tilde{\theta}), R(\hat{\theta}_n^R), R(\hat{\hat{\theta}}) \xrightarrow{p} R(\theta_0)$$