Last time: MLE under misspecification

- $f(y,x;\theta)$ is what we estimate
- $h(y,x;\delta)$ is the true distribution

$\hat{\theta}_n \xrightarrow{p} \theta_q = \operatorname{argmax} E_0[\ln f(y,x;\theta)]$

How does this compare to δ_0?

$$E_0[\ln h(y,x;\delta) - \ln f(y,x;\theta)] = \iint [\ln h(y,x;\delta) - \ln f(y,x;\theta)]\, h(y,x;\delta_0)\, dy\, dx$$
$$\equiv K(h,f)$$

Where $K(h,f)$ defines a metric.

$K(h,f) = 0$ if $h = f$.

It turns out that θ_q minimizes $K(h,f)$. Wow.

$\sqrt{n}(\hat{\theta}_n - \theta_q) \xrightarrow{d} N(0, \Lambda_0)$ what is Λ_0?

$$\frac{\partial \frac{1}{n}\ln L_n(\hat{\theta}_n)}{\partial \theta} = 0 \quad \text{and} \quad E_0\left[\frac{\partial \ln f(x,y;\theta_q)}{\partial \theta}\right] = 0$$

By MVT,

$$\sqrt{n}(\hat{\theta}_n - \theta_q) = \underbrace{\left[-\frac{1}{n}\frac{\partial^2 \ln L_n(\tilde{\theta})}{\partial\theta\partial\theta'}\right]^{-1}}_{(1)} \sqrt{n}\underbrace{\left[\frac{1}{n}\frac{\partial \ln L_n(\theta_q)}{\partial\theta}\right]}_{(2)}$$

where $\tilde{\theta} \in |\hat{\theta}_n, \theta_q|$

Recall:

(1) $\xrightarrow{p} E\left[-\frac{\partial^2 \ln f(x,y;\theta_q)}{\partial\theta\partial\theta'}\right]^{-1} \equiv [A(\theta_q)]^{-1}$

(2) $\xrightarrow{d} N\left(0, \underbrace{E_0\left[\frac{\partial \ln f(x,y;\theta_q)}{\partial\theta}\frac{\partial \ln f(x,y;\theta_q)}{\partial\theta'}\right]}_{\equiv J(\theta_q)}\right)$

$\Rightarrow \sqrt{n}(\hat{\theta}_n - \theta_q) \xrightarrow{d} N(0, [A(\theta_q)]^{-1} J(\theta_q) [A(\theta_q)]^{-1})$

Lecture 10: Generalized Method of Moments

Everything we have learned so far is GMM

$E_0[Y^r]$ can be computed if we know the true distribution.

$E_0[Y^r] \equiv m_r(\theta_0)$ some function of θ_0.

$\frac{1}{n}\sum_{i=1}^{n} Y_i^r = m_r(\hat{\theta}_n)$ where $\hat{\theta}_n$ is the MM estimator for θ_0.

Suppose $f(y;\theta_0) = \theta_0 \exp\{-\theta_0 y\} 1_{\{y \geq 0\}}$

$$E_0[Y] = \int_0^\infty y\theta_0 \exp\{-\theta_0 y\}dy = \frac{1}{\theta_0} \equiv m_1(\theta_0)$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} Y_i = \frac{1}{\hat{\theta}_n} \Rightarrow \hat{\theta}_n = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} Y_i} = \frac{1}{\bar{Y}_r}$$

Dependent variable: Y

Independent random vector: X

$\Big\}\ W_i = \begin{bmatrix} Y_i \\ X_i' \end{bmatrix}$ iid

Vector of parameters: $\theta \in \Theta$

Population parameter θ_0

assume $f(y,x;\theta_0) = f(y|x;\theta_0)\, g(x)$

Is there a nonconstant function φ satisfying

$$E_0[\varphi(y,x;\theta)] = 0 \Longleftrightarrow \theta = \theta_0 ?$$

Define $\varphi(Y, X; \theta) = Y^r - m_r(\theta)$

$$\Rightarrow E_0[Y^r - m_r(\theta)] = 0 \quad \text{iff } \theta = \theta_0 \quad \forall r.$$

We have many equations in only $\dim(\theta)$ unknowns.

$$\min_{b \in B} \frac{1}{n}\sum_{i=1}^{n}(Y_i - X_i' b)^2 \quad \text{gives us } \hat{\beta}_{OLS}$$

FOCs: $$\frac{1}{n}\sum_{i=1}^{n}(Y_i - X_i'\hat{\beta}_{OLS})X_i = 0$$

Here, define $\varphi(Y_i, X_i, \beta) \equiv (Y_i - X_i'\beta)X_i$

$$E_0[\varphi(Y_i, X_i; \beta)] = E_0[(Y_i - X_i'\beta)X_i]$$
$$= E_0[X_i Y_i] - E_0[X_i X_i'\beta] = 0$$

iff $E_0[X_i Y_i] = E_0[X_i X_i']\beta$

iff $\beta = [E_0[X_i X_i']]^{-1} E_0[X_i Y_i] \stackrel{!}{=} \beta_0$.

Thus, $\hat{\beta}_{OLS}$ is a MM estimator.

Generalized Method of Moments

Let $\varphi_j(Y, \theta) = Y^j - m_j(\theta)$

$\Rightarrow E_0[\varphi_1(Y, \theta_0)] = 0 \quad \Rightarrow \quad \frac{1}{n}\sum_{i=1}^{n}(Y_i - m_1(\hat{\theta}_n)) = 0$

$\vdots$

$E_0[\varphi_j(Y, \theta_0)] = 0 \quad\quad \frac{1}{n}\sum_{i=1}^{n}(Y_i^j - m_j(\hat{\theta}_n)) = 0$

assume $m_1(\theta) = \frac{1}{\theta}$, $m_2(\theta) = \frac{1}{\theta^2}$

$$\Rightarrow \left.\begin{aligned} &\frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \frac{1}{\hat{\theta}_n}\right) = 0 \\ &\frac{1}{n}\sum_{i=1}^{n}\left(Y_i^2 - \frac{1}{\hat{\theta}_n^2}\right) = 0 \end{aligned}\right\} \text{There need not be a solution to this system of equations.}$$

Define $\hat{\theta}_n^{GMM} \equiv \arg\min m_n(\theta)' V_n^{-1} m_n(\theta)$

$$= \arg\min \left[\frac{1}{n}\varphi(Y, X; \theta)\right]' V_n^{-1}\left[\frac{1}{n}\varphi(Y, X; \theta)\right]$$

and $V_n \xrightarrow{P} V$ nonsingular nonstochastic