Final 2003 Q5

Let $X_1, \dots, X_n \sim \text{Bernoulli}(p)$

$$f_{X_i}(x_i) = p^{x_i}(1-p)^{(1-x_i)}$$

1] Write the likelihood function

$$L(p \mid X_1, \dots, X_n) = \prod_{i=1}^{n} f_{X_i}(X_i; p)$$

$$= \prod_{i=1}^{n} p^{X_i}(1-p)^{(1-X_i)}$$

$$= p^{\sum_{i=1}^{n} X_i}(1-p)^{\sum_{i=1}^{n}(1-X_i)}$$

2] Find $\hat{p}_{ML}$.

Taking logs, $\log L = \log p \sum_{i=1}^{n} X_i + \log(1-p)\sum_{i=1}^{n}(1-X_i)$

FOCs:

(p): $$\frac{1}{\hat{p}_{ML}}\sum_{i=1}^{n} X_i = \frac{1}{1-\hat{p}_{ML}}\sum_{i=1}^{n}(1-X_i)$$

$$\Rightarrow \sum_{i=1}^{n} X_i - \hat{p}_{ML}\sum_{i=1}^{n} X_i = n\hat{p}_{ML} - \hat{p}_{ML}\sum_{i=1}^{n} X_i$$

$$\Rightarrow \hat{p}_{ML} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

3] Asymptotic distribution:

Recall: $\sqrt{n}(\hat{p}_{ML} - p_0) \xrightarrow{d} N(0, I_1^{-1}(p_0))$

Derivatives are:

$$\frac{\partial \log L}{\partial p} = \frac{1}{p}\sum_{i=1}^{n} X_i - \frac{1}{1-p}\sum_{i=1}^{n}(1-X_i)$$

$$\frac{\partial^2 \log L}{\partial p^2} = -\frac{1}{p^2}\sum_{i=1}^{n} X_i - \frac{1}{(1-p)^2}\sum_{i=1}^{n}(1-X_i)$$

$$I_n(p_0) = E\left[-\frac{\partial^2 \log L}{\partial p^2}\right] = \frac{n p_0}{p_0^2} + \frac{n(1-p_0)}{(1-p_0)^2} = \frac{n}{p_0} + \frac{n}{1-p_0} = \frac{n - np_0 + np_0}{p_0(1-p_0)}$$

$$= \frac{n}{p_0(1-p_0)}$$

$$\Rightarrow I_1(p_0) = \frac{1}{p_0(1-p_0)}$$

$$\Rightarrow I_1^{-1}(p_0) = p_0(1-p_0)$$

Thus, $\sqrt{n}(\hat{p}_{ML} - p_0) \xrightarrow{d} N(0, p_0(1-p_0))$

$\boxed{4}$ What is $[\widehat{p(1-p)}]_{ML}$? ie let $h(p) = p(1-p)$. What is $\hat{h}_{ML}$?

By the invariance principle,

$$[\widehat{p(1-p)}]_{ML} = \hat{p}_{ML}(1-\hat{p}_{ML})$$

$$= \left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]\left[1 - \sum_{i=1}^{n} X_i\right]$$

By the δ-method,

$$\sqrt{n}(h(\hat{p}_{ML}) - h(p_0)) \xrightarrow{d} N\left(0, \left(\frac{dh}{dp}\Big|_{p_0}\right)^2 [p_0(1-p_0)]\right)$$

where $\frac{dh}{dp}\Big|_{p_0} = 1 - 2p_0 \Rightarrow \left(\frac{dh}{dp}\Big|_{p_0}\right)^2 = 1 - 4p_0 + 4p_0^2$

$$\Rightarrow \sqrt{n}(h(\hat{p}_{ML}) - h(p_0)) \xrightarrow{d} N(0, p_0(1-p_0)(1-2p_0)^2)$$

$\boxed{5}$ Consistent estimator for asymptotic covariance matrix

$$\Delta_0 = p_0(1-p_0)(1-2p_0)^2$$

$$\Rightarrow \hat{\Delta}_{ML} = \hat{p}_{ML}(1-\hat{p}_{ML})(1-2\hat{p}_{ML})^2$$

or $\Delta_0 = p_0(1-p_0)$.

$$\Rightarrow \hat{\Delta}_{ML} = \hat{p}_{ML}(1-\hat{p}_{ML})$$

Review Questions, Problem 3

$Y_i = X_i'\beta + u_i$

$u_i = \sigma(X_i)\varepsilon_i$ $\qquad \varepsilon_i | X_i \sim N(0, \sigma_\varepsilon^2)$

where $\sigma(X_i) = 1 + X_i'\gamma$

Provide a consistent estimator for β and derive asymptotic distribution.

1) $\hat\beta_{OLS} = (X'X)^{-1}X'Y = \beta + (X'X)^{-1}X'u$

$= \beta + \left(\frac{1}{n}\sum_{i=1}^n X_i X_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n X_i u_i\right)$

where $\frac{1}{n}\sum_{i=1}^n X_i X_i' \xrightarrow{p} E[X_i X_i']$

$\Rightarrow \left[\frac{1}{n}\sum_{i=1}^n X_i X_i'\right]^{-1} \xrightarrow{p} \left[E[X_i X_i']\right]^{-1}$

and $\frac{1}{n}\sum_{i=1}^n X_i(1 + X_i'\gamma)\varepsilon_i \xrightarrow{p} E[X_i\varepsilon_i] + E[X_i X_i'\gamma\varepsilon_i]$

and $E[X_i\varepsilon_i] = E[E[X_i\varepsilon_i | X_i]] = E[X_i E[\varepsilon_i | X_i]] = 0$

$E[X_i X_i'\gamma\varepsilon_i] = E[E[X_i X_i'\gamma\varepsilon_i | X_i]] = E[X_i X_i'\gamma E[\varepsilon_i | X_i]] = 0$

$\Rightarrow$ By Slutsky's theorem, $\hat\beta_{OLS} \xrightarrow{p} \beta$

2) $\hat\beta_{OLS} - \beta = (X'X)^{-1}X'u$

$\Rightarrow \sqrt{n}(\hat\beta_{OLS} - \beta) = \left(\frac{1}{n}\sum_{i=1}^n X_i X_i'\right)^{-1}\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n X_i u_i\right) \xrightarrow{d} N(0, \Delta)$

where $\Delta = E[X_i X_i']^{-1} V(X_i u_i)\, E[X_i X_i']^{-1}$

and $V(u_i | X_i) = E[X_i X_i' u_i^2]$

$= E[X_i X_i'(1 + X_i'\gamma)^2\varepsilon_i^2]$

$= \sigma_\varepsilon^2 E[X_i X_i'(1 + X_i'\gamma)^2]$

$\Rightarrow \Delta = \sigma_\varepsilon^2 E[X_i X_i']^{-1} E[X_i X_i'(1 + X_i'\gamma)^2] E[X_i X_i']$

3] Is γ identified?

$u_i = \sigma(X_i)\varepsilon_i = (1 + X_i'\gamma)\varepsilon_i$

$E[u_i | X_i] = 0$

$E[u_i^2 | X_i] = E[(1 + X_i'\gamma)^2 \varepsilon_i^2 | X_i] = \sigma_\varepsilon^2 (1 + X_i'\gamma)^2$

$= \sigma_\varepsilon^2 (1 + 2X_i'\gamma + (X_i'\gamma)^2)$ (*)

Suppose X_i is a scalar:

Then (*) is a regression:

$u_i^2 = \sigma_\varepsilon^2 + X_i \, 2\sigma_\varepsilon^2 \gamma + X_i^2 \sigma_\varepsilon^2 \gamma^2 + v_i$

$= \alpha_1 + \alpha_2 X_i + \alpha_3 X_i^2 + v_i$

$\Rightarrow$ Can estimate $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3$ using OLS of $\widehat{(u_i^2)}$ on $1, X_i, X_i^2$

$\Rightarrow \gamma$ is identified since $\hat{\gamma} = \dfrac{\hat{\alpha}_2}{2\hat{\alpha}_1}$

Suppose X_i is not a scalar ie $X_i = \begin{bmatrix} Z_i \\ W_i \end{bmatrix}$

$\Rightarrow u_i^2 = \sigma_\varepsilon^2 + 2Z_i\gamma_1 + 2W_i\gamma_2 + (Z_i\gamma_1 + W_i\gamma_2)^2 + v_i$

$= \sigma_\varepsilon^2 + 2Z_i\gamma_1 + 2W_i\gamma_2 + Z_i^2\gamma_1^2 + W_i^2\gamma_2 + 2Z_iW_i\gamma_1\gamma_2 + v_i$

Can still identify γ_1, γ_2.