By the mean-value theorem,

$$\underbrace{-\frac{1}{n}\frac{\partial^2 \ln L_n(\tilde{\theta}_n)}{\partial\theta\partial\theta'}}_{B}\sqrt{n}(\hat{\theta}_n-\theta_0)=\underbrace{\frac{1}{\sqrt{n}}\frac{\partial \ln L_n(\theta_0)}{\partial\theta}}_{A}$$

$$A=\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{\partial \ln f(y_i,x_i;\theta_0)}{\partial\theta}\right]\xrightarrow{d}N(0,V_0)$$

where $V_0=E_0\left[\frac{\partial \ln f(y_i,x_i;\theta_0)}{\partial\theta}\frac{\partial \ln f(y_i,x_i;\theta_0)}{\partial\theta'}\right]$

$$=I_0(\theta_0)$$

$$B=\frac{1}{n}\sum_{i=1}^{n}\left[-\frac{\partial^2 \ln f(y_i,x_i;\tilde{\theta}_n)}{\partial\theta\partial\theta'}\right]\xrightarrow{p}I_0(\theta_0)$$ To be shown

$$\Rightarrow \sqrt{n}(\hat{\theta}_n-\theta_0)\xrightarrow{d}N(0,[I_0(\theta_0)]^{-1})$$

Now, we have:

$$\frac{\partial \ln f(y,x;\theta)}{\partial\theta}=\frac{1}{f(y,x;\theta)}\frac{\partial f(y,x;\theta)}{\partial\theta}$$

$$\frac{\partial^2 \ln f(y,x;\theta)}{\partial\theta\partial\theta'}=\frac{1}{f(y,x;\theta)}\frac{\partial^2 f(y,x;\theta)}{\partial\theta\partial\theta'}-\frac{1}{[f(y,x;\theta)]^2}\frac{\partial f(y,x;\theta)}{\partial\theta}\frac{\partial f(y,x;\theta)}{\partial\theta'}$$

$$=\frac{1}{f(y,x;\theta)}\frac{\partial^2 f(y,x;\theta)}{\partial\theta\partial\theta'}-\frac{\partial \ln f(y,x;\theta)}{\partial\theta}\frac{\partial \ln f(y,x;\theta)}{\partial\theta'}$$

$$E_0\left[\frac{-\partial^2 \ln f(y,x;\theta)}{\partial\theta\partial\theta'}\right]=-E_0\left[\frac{1}{f(y,x;\theta)}\frac{\partial^2 f(y,x;\theta)}{\partial\theta\partial\theta'}\right]+E_0\left[\frac{\partial \ln f(y,x;\theta_0)}{\partial\theta}\frac{\partial \ln f(y,x;\theta)}{\partial\theta'}\right]$$

$$E_0\left[\frac{1}{f(y,x;\theta_0)}\frac{\partial^2 f(y,x;\theta_0)}{\partial\theta\partial\theta'}\right] = \iint \frac{1}{f(y,x;\theta_0)}\frac{\partial^2 f(y,x;\theta_0)}{\partial\theta\partial\theta'} f(y,x;\theta_0)\,dx\,dy$$

$$= \iint \frac{\partial^2 f(y,x;\theta_0)}{\partial\theta\partial\theta'}\,dx\,dy$$

$$= \frac{\partial^2}{\partial\theta\partial\theta'}\iint f(y,x;\theta_0)\,dx\,dy$$

$$= \frac{\partial^2}{\partial\theta\partial\theta'} 1 = 0$$

$$\Rightarrow E_0\left[-\frac{\partial^2 \ln f(y,x;\theta_0)}{\partial\theta\partial\theta'}\right] = E_0\left[\frac{\partial \ln f(y,x;\theta_0)}{\partial\theta}\frac{\partial \ln f(y,x;\theta_0)}{\partial\theta'}\right] = I_0$$

This gives us that

$$\sqrt{n}(\hat\theta_n - \theta_0) \xrightarrow{d} N(0, I^{-1}(\theta_0))$$

But we do not know $I^{-1}(\theta_0)$. We can estimate it in two ways:

- $\widehat{I(\theta_0)} = \frac{1}{n}\sum_{i=1}^{n} \frac{\partial \ln f(Y_i, X_i; \hat\theta_n)}{\partial\theta}\frac{\partial \ln f(Y_i, X_i; \hat\theta_n)}{\partial\theta'} \xrightarrow{P} I(\theta_0)$
- $\widehat{I(\theta_0)}' = -\frac{1}{n}\sum_{i=1}^{n} \frac{\partial^2 \ln f(Y_i, X_i; \hat\theta_n)}{\partial\theta\partial\theta'} \xrightarrow{P} I(\theta_0)$

Which estimator should we use?

- Computer will always give you the Hessian matrix estimate.

What happens if we have misspecification?
i.e. what if f is not the true pdf?

Suppose $f(y,x;\theta) = f(y;x,\theta)\, g(x)$ is what we use, but the truth is $f(y,x;\theta) = f(y;x,\theta)\, h(x)$ where $g(x) \neq h(x)$.

- ie "oversampling."

This will not be a problem, because when we take logs and FOCs, $h(x)$ and $g(x)$ do not affect the estimate.

Or, we could have the problem where we use $f(y;x,\theta)$, but we should have used $h(y;x,\delta)$

$$\Rightarrow \frac{1}{n}\sum \ln f(y,x;\theta) \xrightarrow{p} E_0[\ln f(y,x;\theta)]$$

$$= \int \log f(y,x;\theta)\, h(y,x;\delta_0)\, dx\, dy$$

How is $\theta_q = \underset{\theta\in\Theta}{\arg\max}\ E_0[\ln f(y,x;\theta)]$ related to δ_0?

Can we still estimate θ_q? Yes: $\hat{\theta}_n = \underset{\theta\in\Theta}{\arg\max}\ \frac{1}{n}\sum \ln f(y_i|x_i,\theta)$

$$\Rightarrow \hat{\theta}_n \xrightarrow{p} \theta_q$$