2003 Spring Comp

Q1: $y_i = \underset{1\times k\ k\times 1}{\beta' x_i} + \underset{1\times p\ p\times 1}{\gamma' w_i} + u_i \qquad (i=1,\dots,n)$

$E[x_i u_i] = 0$

$E[w_i u_i] \neq 0$

a) Can we estimate β consistently using least squares?

- No. Endogeneity even affects the exogenous regressor.

$$\hat{\beta}_{OLS} = (X' M_W X)^{-1} X' M_W Y$$
$$= (X' M_W X)^{-1} X' M_W (X\beta + W\gamma + u)$$
$$= \beta + (X' M_W X)^{-1} X' M_W u$$
$$= \beta + (X'X - X'W(W'W)^{-1}W'X)^{-1}(X'u - X'W(W'W)W'u)$$
$$= \beta + \left[\sum_{i=1}^n X_i X_i' - \left(\sum_{i=1}^n w_i X_i'\right)'\left(\sum_{i=1}^n w_i w_i'\right)^{-1}\left[\sum_{i=1}^n w_i X_i'\right]\right]^{-1}$$
$$\cdot \left[\underbrace{\sum_{i=1}^n X_i u_i}_{\xrightarrow{p} 0} - \left(\sum_{i=1}^n w_i X_i'\right)'\left(\sum_{i=1}^n w_i w_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n w_i u_i\right)\right]$$
$$\xrightarrow{p} \beta - \left[\Sigma_{XX} - \Sigma_{WX}' \Sigma_{WW}^{-1} \Sigma_{WX}\right]^{-1}\left[\Sigma_{WX}' \Sigma_{WW}^{-1} E[w_i u_i]\right]$$
$$= \beta \text{ iff } \Sigma_{WX} = 0.$$

b) Suppose $Cov(x_i, w_i') = 0$ and $X'W = 0$

$(\Leftrightarrow E[x_i w_i'] = 0 \text{ if } E[x_i] = 0 \text{ or } E[w_i'] = 0)$

Let $z_i = \begin{bmatrix} z_{i1} \\ \vdots \\ z_{il} \end{bmatrix}$ be an instrument for w_i

If $E[w_i u_i] = 0$ were true, we could estimate

$$\hat{\gamma}_{OLS} = (W' M_X W)^{-1} W' M_X Y$$
$$= (\hat{W}'\hat{W})^{-1}\hat{W}'Y \qquad \text{where } \hat{W} = M_X W \quad (i)$$
$$= (\hat{W}'\hat{W})^{-1}\hat{W}'\hat{Y} \qquad \text{where } \hat{Y} = M_X Y \quad (ii)$$

(i) Residual regression
- Regress W on X, get residual $\tilde{W}$
- Regress Y on W.

(ii) Double-Residual regression
- Regress W on X, Y on X, get residuals $\tilde{W}$, $\tilde{Y}$
- Regress $\tilde{Y}$ on $\tilde{W}$.

Since $X'W=0$, we will have

$$\hat{\gamma}_{OLS} = (\tilde{W}'\tilde{W})^{-1}\tilde{W}'Y$$
$$= (W'(I-P_X)W)^{-1}W'(I-P_X)Y$$
$$= (W'W - W'\underbrace{P_X W}_{=X(X'X)^{-1}\underbrace{X'W}_{=0}})^{-1}[W'Y - \underbrace{W'P_X Y}_{=0}]$$
$$= (W'W)^{-1}W'Y$$
$$= (W'W)^{-1}W'\tilde{Y}$$

But $\hat{\gamma}_{OLS} \overset{p}{\nrightarrow} \gamma$ by endogeneity. Thus, we will use 2SLS:

$$\hat{\gamma}^{(b)}_{2SLS} = (\hat{W}'M_X\hat{W})^{-1}\hat{W}'M_X Y \quad \text{where } \hat{W} = P_Z W$$

└projection onto instruments.

$$= (W'P_Z'M_X P_Z W)^{-1}W'P_Z'M_X Y$$
$$= (W'P_Z'P_Z W - W'P_Z'P_X P_Z W)^{-1}W'P_Z M_X Y$$
$$= (W'P_Z W - W'P_Z P_X P_Z W)^{-1}W'P_Z M_X Y$$

c) i) Regress Y on $X \Rightarrow \hat{\beta} = (X'X)^{-1}X'Y$

$$\Rightarrow \hat{Y} = M_X Y$$

ii) Regress $\hat{Y}$ on W using Z as instrument.

└2SLS: Regress $\hat{Y}$ on $\hat{W} = P_Z W$.

$$\hat{\gamma}^{(c)}_{2SLS} = (\hat{W}'\hat{W})^{-1}\hat{W}'\hat{Y}$$
$$= (W'P_Z W)^{-1} W'P_Z M_X Y$$

d) Here, we see that $\hat{\gamma}^{(b)}_{2SLS} = \hat{\gamma}^{(c)}_{2SLS}$ only if $W'P_Z P_X P_Z W = 0$, but this need not be true if Z and X are correlated.

$\hat{\gamma}^{(b)}_{2SLS} \xrightarrow{p} \hat{\gamma}^{(c)}_{2SLS}$ if $E[z_i X_i'] = 0$ or

$\hat{\gamma}^{(b)}_{2SLS} = \hat{\gamma}^{(c)}_{2SLS}$ if $Z'X = 0$.

Claim: If $E[X_i v_i] = 0$ where $W_i' = Z_i'\Pi + v_i$, then $E[X_i Z_i'] = 0$

Pf: $\underbrace{E[X_i W_i']}_{=0} = E[X_i Z_i']\Pi + \underbrace{E[X_i v_i]}_{=0}$

$\Rightarrow E[X_i Z_i'] = 0$ (since $\Pi \neq 0$ by assumption that Z is valid instr. for X.)

Thus, if we assume $E[X_i v_i] = 0$, we have that $\hat{\gamma}^{(b)}_{2SLS} \xrightarrow{p} \hat{\gamma}^{(c)}_{2SLS}$.