Convergence in probability almost winning $1M
Convergence almost surely winning almost $1M.

$\sup_\theta |g(W,\theta)| \leq M(W)$ uniform [illegible]

If $E[\sup_\theta |g(W,\theta)|] < +\infty$ we can let

$M(W) = E[\sup_\theta |g(w,\theta)|]$.

Let $g(W,\theta) = \log f(\{Y_i\},\{X_i\};\theta) = \log[f(\{Y_i\};\{X_i\},\theta)\, g(\{X_i\})]$

Is $\sup_\theta |\log[f(\{Y_i\};\{X_i\},\theta) g(\{X_i\})]| < +\infty$?

$$\sup_\theta |\log f(\{Y_i\}|\{X_i\},\theta) + \log g(\{X_i\})|$$

$$\leq \sup_\theta \{|\log f(\{Y_i\}|\{X_i\},\theta)|\} + |\log g(\{X_i\})|$$

Assumption X: $E[|\log g(\{X_i\})|] < +\infty$

Is $E[\sup_\theta |\log f(\{Y_i\}|\{X_i\},\theta)|] < +\infty$?

For simplicity, let $\theta \in \Theta = [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}$ (Θ compact)

Returning to the Logit example

$$\log f(Y_i | X_i, \theta) = Y_i \ln\left(\frac{e^{\theta X_i}}{1+e^{\theta X_i}}\right) + (1-Y_i)\ln\left(\frac{1}{1+e^{\theta X_i}}\right)$$

Is $\sup_{\theta\in[\underline{\theta},\bar{\theta}]} |\log f(Y_i|X_i,\theta)| < +\infty$?

$$\sup_{\theta\in[\underline{\theta},\bar{\theta}]} \left| Y_i \ln\left[\frac{e^{\theta X_i}}{1+e^{\theta X_i}}\right] + (1-Y_i)\ln\left[\frac{1}{1+e^{\theta X_i}}\right]\right|$$

$$= -\inf_{\theta\in[\underline{\theta},\bar{\theta}]} \left| Y_i \ln\left[\frac{e^{\theta X_i}}{1+e^{\theta X_i}}\right] + (1-Y_i)\ln\left[\frac{1}{1+e^{\theta X_i}}\right]\right| \quad (*)$$

Define $I_0(X_i) = I(X_i > 0)$

$$\Rightarrow (*) = -\left\{ Y_i \ln\left[\frac{\exp\{[\underline{\theta} I_0 + \bar{\theta}(1-I_0)]X_i\}}{1+\exp\{[\underline{\theta} I_0 + \bar{\theta}(1-I_0)]X_i\}}\right] + (1-Y_i)\ln\left[\frac{1}{1+\exp\{[\underline{\theta} I_0 + \bar{\theta}(1-I_0)]X_i\}}\right]\right\}$$

$$= -\{Y_i \ln p(X_i) + (1-Y_i)\ln(1-p(X_i))\}$$

$$= -[Y_i(\ln p(X_i) - \ln(1-p(X_i))) + \ln(1-p(X_i))]$$

$$= -\ln(1-p(X_i)) - Y_i \ln\left[\frac{p(X_i)}{1-p(X_i)}\right]$$

$$(**) \quad \le -\ln(1-p(X_i)) - I\left(p(X_i) < \tfrac{1}{2}\right)\ln\left[\frac{p(X_i)}{1-p(X_i)}\right]$$

If $0 < \underline{p} \le p(X_i) \le \bar{p} < 1 \ \forall X_i$, then $(**)$ is bounded in X_i.

$$\Rightarrow E\left[\sup_{\theta\in[\underline{\theta},\bar{\theta}]} |\log f(Y_i | X_i, \theta)|\right] < +\infty$$

Typically, we just assume

Assumption SUP: $E_0\left[\sup_{\theta\in\Theta} |\log f(\{Y_i\} | \{X_i\}, \theta)|\right] < +\infty$

Finally, we have

$$\frac{1}{n} \log L(\theta; \{X_i\}, \{Y_i\}) = \frac{1}{n} \sum_{i=1}^{n} \log f(Y_i, X_i; \theta)$$
$$\xrightarrow{a.s.} E_0 [\log f(Y_i, X_i; \theta)]$$

uniformly in θ.

$$E_0 [\log f(Y_i, X_i; \theta)] = \iint f(y_i, x_i; \theta) f(y_i, x_i; \theta_0) dx_i dy_i$$