Simultaneous Equations model and Instrumental Variables.

SEM: (Greene, p. 390)

$y_{ji} = x_{ji}\beta_j + \varepsilon_{ji}$ $\quad j=1,\dots,M$ (ie M endogenous variables)

$$\underset{1\times1}{y_{ji}} = \underset{1\times M_j}{\underline{Y}_{ji}'}\gamma_j + \underset{1\times M_j^*}{\underline{Y}_{ji}^{*\prime}}\gamma_j^* + \underset{1\times K_j}{x_{ji}'}\beta_j + \underset{1\times K_j^*}{x_{ji}^{*\prime}}\beta_j^* + \varepsilon_{ji} \qquad (*)$$

where $M_j + M_j^* + 1 = M$, $K_j + K_j^* = K$

γ_j^* contains the parameters we have restricted in the j^{th} equation for endogenous variables

β_j^* contains the parameters we have restricted in the j^{th} equation for exogenous variables

	endogenous		exogenous	
included	$\underline{Y}_{ji}$	M_j	x_{ji}	K_j
excluded	$\underline{Y}_{ji}^*$	M_j^*	x_{ji}^*	K_j^*

$\gamma_j^* \text{"="} \beta_j^* = 0$

Rewriting $(*)$

$$\underset{1\times M}{\underline{Y}_i'}\Gamma_j + \underset{1\times K}{\underline{X}_i'}B_j = \varepsilon_{ij}$$

where $\underline{Y}_{ji} = \begin{bmatrix} Y_{1i} \\ \vdots \\ Y_{Mi} \end{bmatrix} = \begin{bmatrix} y_{ji} \\ \underline{Y}_{ji} \\ \underline{Y}_{ji}^* \end{bmatrix}$

$\Gamma_j = \begin{bmatrix} 1 \\ -\gamma_j \\ -\gamma_j^* \end{bmatrix} = \begin{bmatrix} 1 \\ -\gamma_j \\ 0 \end{bmatrix}$ imposing the restriction

$\underline{X}_{ji} = \begin{bmatrix} x_{ji} \\ x_{ji}^* \end{bmatrix}$

$B_j = \begin{bmatrix} -\beta_j \\ -\beta_j^* \end{bmatrix} = \begin{bmatrix} -\beta_j \\ 0 \end{bmatrix}$

Converting to reduced form: (we can run this regression)

$$[\underset{1\times 1}{y_{ji}'}\ \ \underset{1\times M_j}{Y_{ji}'}\ \ \underset{1\times M_j^*}{Y_{ji}^{*\prime}}] = [\underset{1\times K_j}{x_{ji}'}\ \ \underset{1\times K_j^*}{x_{ji}^{*\prime}}] \underset{K\times M}{\begin{array}{c} \begin{bmatrix} \pi_j & \underline{\Pi}_j & \overline{\Pi}_j \\ \pi_j^* & \underline{\Pi}_j^* & \overline{\Pi}_j^* \end{bmatrix} \end{array}} \begin{array}{c} K_j \\ K_j^* \end{array} + [\underset{1\times 1}{v_{ji}}\ \ \underset{1\times M_j}{V_{ji}}\ \ \underset{1\times M_j^*}{V_{ji}^*}]$$

where $\underset{K\times 1}{\begin{bmatrix} \pi_j \\ \pi_j^* \end{bmatrix}}$ $\underset{K\times M_j}{\begin{bmatrix} \underline{\Pi}_j \\ \underline{\Pi}_j^* \end{bmatrix}}$ $\underset{K\times M_j^*}{\begin{bmatrix} \overline{\Pi}_j \\ \overline{\Pi}_j^* \end{bmatrix}}$

How do we relate Π_j to B_j and Γ_j?

Let $\Pi = [\Pi_1 \cdots \Pi_M]$, $B = [B_1 \cdots B_M]$, $\Gamma = [\Gamma_1 \cdots \Gamma_M]$

$\Rightarrow \Pi_j = -B_j \Gamma_j^{-1}$

$\Rightarrow -B_j = \Pi_j \Gamma_j$

This gives us:

$$\begin{bmatrix} B_j \\ 0 \end{bmatrix} = \begin{bmatrix} \pi_j & \underline{\Pi}_j & \overline{\Pi}_j \\ \pi_j^* & \underline{\Pi}_j^* & \overline{\Pi}_j^* \end{bmatrix} \begin{bmatrix} 1 \\ -\gamma_j \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad B_j = \pi_j - \underline{\Pi}_j \gamma_j \qquad (1)$$

$$0 = \underset{K_j^*\times 1}{\pi_j^*} - \underset{K_j^*\times M_j}{\underline{\Pi}_j^*} \underset{M_j\times 1}{\gamma_j} \qquad (2)$$

Identification

In (2), we have K_j^* equations and M_j unknowns.

For identification, must have $K_j^* \geq M_j$ (order condition)

(# excluded exogenous regressors $\geq$ # included endogenous regressors)

Instrumental Variables

Trianglular SEM (scalars)

$y = x\beta + z_1\delta_1 + z_2\delta_2 + \varepsilon$ $\qquad E[\varepsilon x] \neq 0$

$x = z_1\pi_1 + z_2\pi_2 + u$ $\qquad E[\varepsilon z] = 0$

$\qquad E[uz] = 0$

$$\Rightarrow y = (\beta\pi_1 + \delta_1)z_1 + (\beta\pi_2 + \delta_2)z_2 + \underbrace{(\beta u + \varepsilon)}_{=\varepsilon^*}$$

This regression makes sense:

OLS gives us $\widehat{\beta\pi_1 + \delta_1}$ and $\widehat{\beta\pi_2 + \delta_2}$

Typically, we are interested in β. We cannot determine this at this point.

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(1) $y = x\beta + z_1\delta_1 + z_2\delta_2 + \varepsilon$

(2) $x = z_1\pi_1 + z_2\pi_2 + z_3\pi_3 + u$

1 included endogenous regressor
1 excluded exogenous regressor
$\Rightarrow$ equation one is identified.

(3) $\Rightarrow y = (\beta\pi_1 + \delta_1)z_1 + (\beta\pi_2 + \delta_2)z_2 + \beta\pi_3 z_3 + (\beta u + \varepsilon)$

OLS gives us $\widehat{\beta\pi_1 + \delta_1}$, $\widehat{\beta\pi_2 + \delta_2}$, and $\widehat{\beta\pi_3}$

also, we can estimate the second equation to get, with OLS, $\hat{\pi}_1$, $\hat{\pi}_2$, $\hat{\pi}_3$

Combining these, $\hat{\beta} = \dfrac{\widehat{\beta\pi_3}}{\hat{\pi}_3} \xrightarrow[\text{Slutsky}]{} \dfrac{\beta\pi_3}{\pi_3} = \beta$

In general

$$\underset{n\times1}{Y} = \underset{n\times k}{X}\,\underset{k\times1}{\beta} + \underset{n\times1}{\varepsilon}$$

$$\underset{n\times k}{X} = \underset{n\times \ell}{Z}\,\underset{\ell\times k}{\Pi} + \underset{n\times k}{V}$$

Two stage least squares (2SLS)

$\boxed{1}$ $\underset{l\times k}{\hat{\Pi}} = (\underset{l\times n}{Z'}\underset{n\times l}{Z})^{-1}\underset{l\times n}{Z'}\underset{n\times k}{X} \Rightarrow \hat{X} = \underset{n\times l}{Z}\underset{l\times k}{\hat{\Pi}} = \underbrace{Z(Z'Z)^{-1}Z'}_{P_Z}X$

$\boxed{2}$ Regress Y on $\hat{X}$

$$\hat{\beta}_{2SLS} = (\underset{k\times n}{\hat{X}'}\underset{n\times k}{\hat{X}})^{-1}\underset{k\times n}{\hat{X}'}\underset{n\times 1}{Y}$$

$$= (X' P_Z X)^{-1} X' P_Z Y$$

Regress Y on the projections of X onto Z.