Econ 203c, Problem Set 3, Spring 2003

1)
$\widehat{\beta}_{iv} = (\widehat{\Pi}' Z'X)^{-1}\widehat{\Pi}' Z'y$

1) Assume $\widehat{\Pi} \overset{p}{\to} \Pi_0$
$\sqrt{n}\,(\widehat{\beta}_{iv} - \beta) \overset{d}{\to} N(0, \Lambda_0)$

where $\Lambda_0 = (\Pi_0'\Sigma_{zx})^{-1}\Pi_0' V_0 \Pi_0 (\Pi_0'\Sigma_{zx})^{-1}$,
plim1/n$Z'X = \Sigma_{zx}$,
and by CLT $1/\sqrt{n}\sum z_i\epsilon_i \overset{d}{\to} N(0, V_0)$

2) Assume $\widehat{\Pi} \overset{p}{\to} V_0^{-1}\Sigma_{zx}$
$\sqrt{n}\,(\widehat{\beta}_{iv} - \beta) \overset{d}{\to} N(0, \Lambda^*)$

where $\Lambda^* = (\Sigma_{zx}' V_0^{-1}\Sigma_{zx})^{-1}\Sigma_{zx}' V_0^{-1} V_0 V_0^{-1}\Sigma_{zx}(\Sigma_{zx}' V_0^{-1}\Sigma_{zx})^{-1}$
Simplify: $\Lambda^* = (\Sigma_{zx}' V_0^{-1}\Sigma_{zx})^{-1}$

3) Show that the asymptotic covariance in 1) is at least as large as the asymptotic covariance in 2).
This implies $(\Lambda_0 - \Lambda^*)$ is positive definite. (Actually, "at least as large" implies positive semi-definite).
Show $(\Lambda_0 - \Lambda^*) = (\Pi_0'\Sigma_{zx})^{-1}\Pi_0' V_0 \Pi_0 (\Pi_0'\Sigma_{zx})^{-1} - (\Sigma_{zx}' V_0^{-1}\Sigma_{zx})^{-1} \geq 0$

To make this proof easier, first normalize the probability limit of the weight matrix, Π_0. Without loss of generality, normalize $\Pi_0' = \Sigma_{zx}'\Phi^{-1}$ where Φ is a positive definite, invertible matrix. Thus,
$(\Lambda_0 - \Lambda^*) = (\Sigma_{zx}'\Phi^{-1}\Sigma_{zx})^{-1}\Sigma_{zx}'\Phi^{-1} V_0 \Phi^{-1}\Sigma_{zx}(\Sigma_{zx}'\Phi^{-1}\Sigma_{zx})^{-1} - (\Sigma_{zx}' V_0^{-1}\Sigma_{zx})^{-1}$

Note that for any two matrices A and B, $(A - B) \geq 0$ iff $(B^{-1} - A^{-1}) \geq 0$.
Thus, $(\Lambda_0 - \Lambda^*) \geq 0$

iff $\Sigma_{zx}' V_0^{-1}\Sigma_{zx} - \Sigma_{zx}'\Phi^{-1}\Sigma_{zx}(\Sigma_{zx}'\Phi^{-1} V_0 \Phi^{-1}\Sigma_{zx})^{-1}\Sigma_{zx}'\Phi^{-1}\Sigma_{zx} \geq 0$

$\Sigma_{zx}' V_0^{-1}\Sigma_{zx} - \Sigma_{zx}'\Phi^{-1}\Sigma_{zx}(\Sigma_{zx}'\Phi^{-1} V_0 \Phi^{-1}\Sigma_{zx})^{-1}\Sigma_{zx}'\Phi^{-1}\Sigma_{zx}$

$= \Sigma_{zx}' V_0^{-1/2} [I - V_0^{1/2} \Phi^{-1} \Sigma_{zx} (\Sigma_{zx}' \Phi^{-1} V_0 \Phi^{-1} \Sigma_{zx})^{-1} \Sigma_{zx}' \Phi^{-1} V_0^{1/2}] V_0^{-1/2} \Sigma_{zx}$
$= HPH'$
where $H = \Sigma_{zx}' V_0^{-1/2}$ and $P = [I - V_0^{1/2} \Phi^{-1} \Sigma_{zx} (\Sigma_{zx}' \Phi^{-1} V_0 \Phi^{-1} \Sigma_{zx})^{-1} \Sigma_{zx} \Phi^{-1} V_0^{1/2}]$
Note that P is a projection matrix and idempotent ($P'P = P$).

Thus, $(\Lambda_0 - \Lambda^*) = HPH' = HP(HP)' \geq 0$
$HP(HP)'$ is positive semi-definite.

Question 2
1)-5) See attached Matlab code and output.

6) GLS and OLS coefficients slightly different.

7) Standard errors for GLS are smaller than OLS standard errors.

8) The easiest way to solve for restricted estimators in this SUR framework is to rewrite the X matrix as

$$X = \begin{bmatrix} x1 & x2 & x3 & x4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x5 & x6 & x7 & x8 & 0 & 0 \\ x1 & x2 & 0 & 0 & x5 & x6 & 0 & 0 & x9 & x10 \end{bmatrix}$$

Then the OLS estimates are $\beta = (X'X)^{-1}X'Y$.

You could also compute GLS estimates by using this first stage OLS estimates to compute the residuals from the system and estimating the restricted covariance matrix.

To see why this restricted X matrix implies the restrictions, let's consider a much simpler case.

$y_1 = \beta_{1,1} x_1 + \epsilon_1$

$y_2 = \beta_{2,1}x_1 + \epsilon_2$

The restriction is $\beta_{1,1} = \beta_{2,1}$.

The restricted X is therefore:

$$X_r = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

Contrast this to the general, unrestricted SUR setup:

$$X_u = \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix}$$

OLS estimates for the restricted model:
$\beta_{ols,r} = (X_r'X_r)^{-1}X_r'Y$ is simply a scalar implying $\beta_{1,1,ols} = \beta_{2,1,ols}$.

Whereas $\beta_{ols,u} = (X_u'X_u)^{-1}X_u'Y$ is 2x1 implying two potentially distinct parameters.

Essentially the restriction decreases the amount of exogenous variation in the system, which if there were endogenous regressors, could make identification more difficult.

9) The original, unrestricted, system:

$y_1 = \beta_{1,1}x_1 + \beta_{1,2}x_2 + \beta_{1,3}x_3 + \beta_{1,4}x_4 + \epsilon_1$

$y_2 = \beta_{2,5}x_5 + \beta_{2,6}x_6 + \beta_{2,7}x_7 + \beta_{2,8}x_8 + \epsilon_2$

$y_2 = \beta_{3,1}x_1 + \beta_{3,2}x_2 + \beta_{3,5}x_5 + \beta_{3,6}x_6 + \beta_{3,9}x_9 + \beta_{3,10}x_{10} + \epsilon_3$

4 Restrictions: $\beta_{1,1} = \beta_{3,1}$, $\beta_{1,2} = \beta_{3,2}$, $\beta_{2,5} = \beta_{3,5}$, $\beta_{2,6} = \beta_{3,6}$.

H_0 : $\beta_{1,1} - \beta_{3,1} = 0, \beta_{1,2} - \beta_{3,2} = 0$, $\beta_{2,5} - \beta_{3,5} = 0$, and $\beta_{2,6} - \beta_{3,6} = 0$
H_a : $\beta_{1,1} - \beta_{3,1} \neq 0, \beta_{1,2} - \beta_{3,2} \neq 0$, $\beta_{2,5} - \beta_{3,5} \neq 0$, and $\beta_{2,6} - \beta_{3,6} \neq 0$

Form an F-test:

$$F_{stat} = (R\beta_{ols} - q)'[R(\hat{\sigma}^2(X'X)^{-1}R']^{-1}(R\beta_{ols} - q)$$

where $\hat{\sigma}^2$ is the estimated homoskedastic variance, X is the original, unrestricted big X,

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix},$$

and $q = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$F_{stat} \sim F[4, 500]$

Result: $F_{stat} = 0.0616$

$F_{95\%} = 2.37$

Fail to reject null hypothesis.

Question 3:

(I used Matlab for all of the calculations)

1) OLS

Eq.1: Let $\alpha_1 = [\gamma_1 \ \beta_{11}]'$ and $X_1 = [y_2 \ x_1]$

$$\alpha_{1,ols} = (X_1'X_1)^{-1}X_1'y_1$$

$$= \begin{bmatrix} y_2'y_2 & y_2'x_1 \\ x_1'y_2 & x_1'x_1 \end{bmatrix}^{-1} \begin{bmatrix} y_2'y_1 \\ x_1'y_1 \end{bmatrix} = \begin{bmatrix} 10 & 3 \\ 3 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4390 \\ 0.5366 \end{bmatrix}$$

Eq.2: Let $\alpha_2 = [\gamma_2 \ \beta_{22} \ \beta_{32}]'$ and $X_2 = [y_1 \ x_2 \ x_3]$

$$\alpha_{2,ols} = (X_2'X_2)^{-1}X_2'y_2$$

$$= \begin{bmatrix} y_1'y_1 & y_1'x_2 & y_1'x_3 \\ x_2'y_1 & x_2'x_2 & x_2'x_3 \\ x_3'y_1 & x_3'x_2 & x_3'x_3 \end{bmatrix}^{-1} \begin{bmatrix} y_1'y_2 \\ x_2'y_2 \\ x_3'y_2 \end{bmatrix} = \begin{bmatrix} 20 & 3 & 5 \\ 3 & 10 & 8 \\ 5 & 8 & 15 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 6 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1930 \\ 0.3841 \\ 0.1975 \end{bmatrix}$$

2) 2sls

Eq.1:
Instruments for y_2 and x_1: $Z_1 = [x_1 \ x_2 \ x_3]$, $X_1 = [y_2 \ x_1]$

$$\alpha_{1,2sls} = [(Z_1(Z_1'Z_1)^{-1}Z_1'X_1)'X_1]^{-1}(Z_1(Z_1'Z_1)^{-1}Z_1'X_1)'y_1$$

$$= [X_1'Z_1(Z_1'Z_1)^{-1}Z_1'X_1]^{-1}X_1'Z_1(Z_1'Z_1)^{-1}Z_1'y_1$$

$$X_1'Z_1 = \begin{bmatrix} y_2'x_1 & y_2'x_2 & y_2'x_3 \\ x_1'x_1 & x_1'x_2 & x_1'x_3 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 7 \\ 5 & 2 & 3 \end{bmatrix} = (Z_1'X_1)'$$

$$(Z_1'Z_1) = \begin{bmatrix} x_1'x_1 & x_1'x_2 & x_1'x_2 \\ x_2'x_1 & x_2'x_2 & x_2'x_3 \\ x_3'x_1 & x_3'x_2 & x_3'x_3 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{bmatrix}$$

$$Z_1'y_1 = \begin{bmatrix} x_1'y_1 \\ x_2'y_1 \\ x_3'y_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

$$\alpha_{1,2sls} = \begin{bmatrix} 0.3688 \\ 0.5787 \end{bmatrix}$$

Eq.2:

Instruments for y_1,x_2, and x_3 : $Z_2 = [x_1\ x_2\ x_3](= Z_1)$, $X_2 = [y_1\ x_2\ x_3]$

$\alpha_{2,2sls} = [X_2'Z_1(Z_1'Z_1)^{-1}Z_1'X_2]^{-1}X_2'Z_1(Z_1'Z_1)^{-1}Z_1'y_2$

$$X_2'Z_1 = \begin{bmatrix} y_1'x_1 & y_1'x_2 & y_1'x_3 \\ x_2'x_1 & x_2'x_2 & x_2'x_3 \\ x_3'x_1 & x_3'x_2 & x_3'x_3 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 5 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{bmatrix} = (Z_1'X_2)'$$

$$Z_1'y_2 = \begin{bmatrix} x_1'y_2 \\ x_2'y_2 \\ x_3'y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 7 \end{bmatrix}$$

$$\alpha_{2,2sls} = \begin{bmatrix} 0.4844 \\ 0.3672 \\ 0.1094 \end{bmatrix}$$

Covariance Matrix

Eq. 1:

$V(\alpha_{1,2sls}) = \Psi_1 V(Z_1'\varepsilon_1)\Psi_1'$

where $\Psi_1 = [X_1'Z_1(Z_1'Z_1)^{-1}Z_1'X_1]^{-1}X_1'Z_1(Z_1'Z_1)^{-1}$

Under the assumption of homoskedasticity, $V(Z_1'\varepsilon_1) = \sigma^2 Z_1'Z_1$.

And $V(\alpha_{1,2sls})$ simplifies: $V(\alpha_{1,2sls}) = \sigma_1^2[X_1'Z_1(Z_1'Z_1)^{-1}Z_1'X_1]^{-1}$

Estimate σ_1^2: $\widehat{\sigma}_1^2 = (1/n)e_1'e_1 = (1/n)(y_1 - X_1\alpha_{1,2sls})'(y_1 - X_1\alpha_{1,2sls})$

$= (1/n)[y_1'y_1 - \alpha_{1,2sls}'X_1'y_1 - y_1'X_1\alpha_{1,2sls} + \alpha_{1,2sls}'X_1'X_1\alpha_{1,2sls}]$

$(1/n)e_1'e_1 = 0.6104$

$$V(\alpha_{1,2sls}) = [X_1'Z_1(Z_1'Z_1)^{-1}Z_1'X_1]^{-1} = \begin{bmatrix} 0.2151 & -0.1290 \\ -0.1290 & 0.1995 \end{bmatrix}$$

Eq. 2:

$$V(\alpha_{2,2sls}) = \hat{\sigma}_2^2[X_2'Z_1(Z_1'Z_1)^{-1}Z_2'X_1]^{-1}$$
$$\hat{\sigma}_2^2 = (1/n)[y_2'y_2 - \alpha_{2,2sls}'X_2'y_2 - y_2'X_2\alpha_{2,2sls} + \alpha_{2,2sls}'X_2'X_2\alpha_{2,2sls}] = 0.2684$$

$$V(\alpha_{2,2sls}) = \begin{bmatrix} 0.1324 & -0.0077 & -0.0400 \\ -0.0077 & 0.0473 & -0.0226 \\ -0.0400 & -0.0226 & 0.0433 \end{bmatrix}$$

3) LIML for eq.1

There is a procedure in Greene (ch. 15) to calculate this. LIML is asymptotically equivalent to 2sls. LIML is rarely, if ever, more efficient than 2sls, which does not rely on a normality assumption. In addition, LIML is a pain to compute.

4) 3sls (Generalized 2sls)

We first need to estimate the covariance matrix using the residuals from the two equations estimated with 2sls.

$$\hat{\Sigma} = \begin{bmatrix} (1/n)e_1'e_1 & (1/n)e_1'e_2 \\ (1/n)e_2'e_1 & (1/n)e_2'e_2 \end{bmatrix}$$

From above: $(1/n)e_1'e_1 = 0.6104$ and $(1/n)e_1'e_1 = 0.2684$.

$$(1/n)e_1'e_2 = (1/n)e_2'e_1 = (1/n)(y_1 - X_1\alpha_{1,2sls})'(y_2 - X_2\alpha_{2,2sls})$$
$$= [y_1'y_2 - \alpha_{1,2sls}'X_1'y_2 - y_1'X_2\alpha_{2,2sls} + \alpha_{1,2sls}'X_1'X_2\alpha_{2,2sls}]$$

$= -0.2743$

Define: $X = \begin{bmatrix} y_2 & x_1 & 0 & 0 & 0 \\ 0 & 0 & y_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} X1 & \ldots & 0 \\ 0 & \ldots & X2 \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$,

$Z = \begin{bmatrix} x_1 & x_2 & x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} Z1 & \ldots & 0 \\ 0 & \ldots & Z2 \end{bmatrix}$, $V = \widehat{\Sigma} \otimes I_n$, and $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$

$\alpha_{3sls} = [\widehat{X}' V^{-1} X]^{-1} \widehat{X}' V^{-1} Y$ where $\widehat{X}' = X' Z (Z'Z)^{-1} Z'$

$V^{-1} = \widehat{\Sigma}^{-1} \otimes I_n,\ \widehat{\Sigma}^{-1} = \begin{bmatrix} \widehat{\sigma}^{11} & \widehat{\sigma}^{12} \\ \widehat{\sigma}^{12} & \widehat{\sigma}^{22} \end{bmatrix}$

$$\widehat{X}' V^{-1} X = \begin{bmatrix} \widehat{\sigma}^{11} \widehat{X_1}' X_1 & \widehat{\sigma}^{12} \widehat{X_1}' X_2 \\ \widehat{\sigma}^{12} \widehat{X_2}' X_1 & \widehat{\sigma}^{22} \widehat{X_2}' X_2 \end{bmatrix}$$

$$= \begin{bmatrix} \widehat{\sigma}^{11} X_1' Z_1 (Z_1' Z_1)^{-1} Z_1' X_1 & \widehat{\sigma}^{12} X_1' Z_1 (Z_1' Z_1)^{-1} Z_1' X_2 \\ \widehat{\sigma}^{12} X_2' Z_1 (Z_1' Z_1)^{-1} Z_1' X_1 & \widehat{\sigma}^{22} X_2' Z_1 (Z_1' Z_1)^{-1} Z_1' X_2 \end{bmatrix}$$

$$\widehat{X}' V^{-1} Y = \begin{bmatrix} \widehat{\sigma}^{11} X_1' Z_1 (Z_1' Z_1)^{-1} Z_1' y_1 \\ \widehat{\sigma}^{22} X_2' Z_1 (Z_1' Z_1)^{-1} Z_1' y_2 \end{bmatrix}$$

$$\alpha_{3sls} = \begin{bmatrix} 0.3688 \\ 0.5787 \\ 0.4715 \\ 0.3104 \\ 0.1644 \end{bmatrix}$$

5) Indirect OLS

Reduced Form:

$y_1 = \gamma_1 y_2 + \beta_{11} x_1 + \varepsilon_1 \qquad (1)$

$y_2 = \gamma_2 y_1 + \beta_{22} x_2 + \beta_{32} x_3 + \varepsilon_2 \qquad (2)$

Substitute 1) into 2):

$$y_2 = \gamma_2(\gamma_1 y_2 + \beta_{11}x_1 + \varepsilon_1) + \beta_{22}x_2 + \beta_{32}x_3 + \varepsilon_2$$

Rearrange:

$$y_2[1 - \gamma_2\gamma_1] = \gamma_2\beta_{11}x_1 + \beta_{22}x_2 + \beta_{32}x_3 + (\varepsilon_2 + \gamma_2\varepsilon_1)$$

$$\begin{aligned} y_2 &= \frac{\gamma_2\beta_{11}}{1-\gamma_2\gamma_1}x_1 + \frac{\beta_{22}}{1-\gamma_2\gamma_1}x_2 + \frac{\beta_{32}}{1-\gamma_2\gamma_1}x_3 + (\varepsilon_2 + \gamma_2\varepsilon_1) \\ &= \delta_{11}x_1 + \delta_{12}x_2 + \delta_{13}x_3 + \eta_1 \end{aligned}$$

$$y_1 = \gamma_1[\frac{\gamma_2\beta_{11}}{1-\gamma_2\gamma_1}x_1 + \frac{\beta_{22}}{1-\gamma_2\gamma_1}x_2 + \frac{\beta_{32}}{1-\gamma_2\gamma_1}x_3 + (\varepsilon_2 + \gamma_2\varepsilon_1)] + \beta_{11}x_1 + \varepsilon_1$$

$$\begin{aligned} y_1 &= [\frac{\gamma_1\gamma_2\beta_{11}}{1-\gamma_2\gamma_1} + \beta_{11}]x_1 + \frac{\gamma_1\beta_{22}}{1-\gamma_2\gamma_1}x_2 + \frac{\gamma_1\beta_{32}}{1-\gamma_2\gamma_1}x_3 + (\varepsilon_1 + \gamma_1\varepsilon_2 + \gamma_1\gamma_2\varepsilon_1) \\ &= \delta_{21}x_1 + \delta_{22}x_2 + \delta_{23}x_3 + \eta_2 \end{aligned}$$

Define: $\delta_1 = [\delta_{11}\ \delta_{12}\ \delta_{13}]$, $\delta_2 = [\delta_{21}\ \delta_{22}\ \delta_{23}]$, $Z = [x_1\ x_2\ x_3]$

OLS estimation of reduced form parameters:

$$\delta_{1,ols} = (Z'Z)^{-1}Z'y_1 = \begin{bmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0.6809 \\ 0.0106 \\ 0.1915 \end{bmatrix}$$

$$\delta_{2,ols} = (Z'Z)^{-1}Z'y_2 = \begin{bmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} 0.3298 \\ 0.3298 \\ 0.2021 \end{bmatrix}$$

Solving for structural parameters:

$$0.6809 = \frac{\gamma_2\beta_{11}}{1-\gamma_2\gamma_1} \qquad (1)$$

$$0.0106 = \frac{\beta_{22}}{1-\gamma_2\gamma_1} \qquad (2)$$

$$0.1915 = \frac{\beta_{32}}{1-\gamma_2\gamma_1} \qquad (3)$$

$$0.3298 = \frac{\gamma_1\gamma_2\beta_{11}}{1-\gamma_2\gamma_1} + \beta_{11} \qquad (4)$$

$$0.3298 = \frac{\gamma_1 \beta_{22}}{1 - \gamma_2 \gamma_1} \qquad (5)$$

$$0.2021 = \frac{\gamma_1 \beta_{32}}{1 - \gamma_2 \gamma_1} \qquad (6)$$

Note: This system is over-identified: there are 6 equations and only 5 unknown structural parameters. Equation 1 is over-identified and Equation 2 is just identified. Because the system is over-identified, we cannot solve for a unique estimate of the structural parameters.