Problem Set 2, Suggested Answers

1)

a)

property 1)

Let A be mxk, B be nxl, C be kxw, D be lxj

$(A \otimes B)(C \otimes D) = AC \otimes BD$

$$A = \begin{bmatrix} a_{11} & \dots & a_{1k} \\ : & & : \\ a_{m1} & \dots & a_{mk} \end{bmatrix} \quad C = \begin{bmatrix} c_{11} & \dots & c_{1w} \\ : & & : \\ c_{k1} & \dots & c_{kw} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & \dots & b_{1l} \\ : & & : \\ b_{n1} & \dots & b_{nl} \end{bmatrix} \quad D = \begin{bmatrix} d_{11} & \dots & d_{1j} \\ : & & : \\ d_{l1} & \dots & d_{lj} \end{bmatrix}$$

$$(A \otimes B) = \begin{bmatrix} a_{11}B & \dots & a_{1k}B \\ : & & : \\ a_{m1}B & \dots & a_{mk}B \end{bmatrix} \quad (C \otimes D) = \begin{bmatrix} c_{11}D & \dots & c_{1w}D \\ : & & : \\ c_{k1}D & \dots & c_{kw}D \end{bmatrix}$$

$$(A \otimes B)(C \otimes D) = \begin{bmatrix} a_{11}B & \dots & a_{1k}B \\ : & & : \\ a_{m1}B & \dots & a_{mk}B \end{bmatrix} \begin{bmatrix} c_{11}D & \dots & c_{1w}D \\ : & & : \\ c_{k1}D & \dots & c_{kw}D \end{bmatrix}$$

1,1 element of $(A \otimes B)(C \otimes D)$ is

$$a_{11}Bc_{11}D + \dots + a_{1k}Bc_{k1}D = a_{11}c_{11}BD + \dots + a_{1k}c_{k1}BD = \sum_{i=1}^{k} a_{1i}c_{i1}BD$$

$$AC = \begin{bmatrix} a_{11} & \dots & a_{1k} \\ : & & : \\ a_{m1} & \dots & a_{mk} \end{bmatrix} \begin{bmatrix} c_{11} & \dots & c_{1w} \\ : & & : \\ c_{k1} & \dots & c_{kw} \end{bmatrix}$$

1,1 element of AC is $\sum_{i=1}^{k} a_{1i}c_{i1}$

1,1 element of $AC \otimes BD = \sum_{i=1}^{k} a_{1i} c_{i1} BD$

Thus, 1,1 element of $(A \otimes B)(C \otimes D)$=1,1 element of $AC \otimes BD$.
Can apply same argument to all elements and thereby prove all elements of

$(A \otimes B)(C \otimes D)$ and $AC \otimes BD$ are the same.

property 2)

$(A \otimes B)' = A' \otimes B'$

$$(A \otimes B) = \begin{bmatrix} a_{11}B & \dots & a_{1k}B \\ \vdots & & \vdots \\ a_{m1}B & \dots & a_{mk}B \end{bmatrix}$$

1,1 block of $(A \otimes B)'$ is $(a_{11}B)' = B'a_{11}$

$$A' \otimes B' = \begin{bmatrix} a_{11} & \dots & a_{m1} \\ \vdots & & \vdots \\ a_{1k} & \dots & a_{mk} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & \dots & b_{n1} \\ \vdots & & \vdots \\ b_{1l} & \dots & b_{nl} \end{bmatrix} = \begin{bmatrix} a_{11}B' & \dots & a_{m1}B' \\ \vdots & & \vdots \\ a_{1k}B' & \dots & a_{mk}B' \end{bmatrix}$$

1,1 block of $A' \otimes B'$ is $(a_{11}B)' = B'a_{11}$ =1,1 block of $(A \otimes B)'$

Same argument for all other blocks.

property 3)

Let A be nxn, B mxm

$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

For any invertible matrices C and D,
if $CD^{-1} = I$ then $C^{-1} = D^{-1}$

Thus, $(A \otimes B)(A^{-1} \otimes B^{-1}) = AA^{-1} \otimes BB^{-1}$ by property shown in 1)

$AA^{-1} \otimes BB^{-1} = I_n \otimes I_m = I_{nm}$

b)
$y = (I \otimes X)\pi + u$
$E[u|X] = 0$ and $V(u|X) = \Sigma \otimes I$

1) Show
$\widehat{\pi}_{gls} = (I \otimes (X'X)^{-1}X')y$

This is the case of identical regressors (see lecture notes)
Let $X^* = (I \otimes X)$, then $y = X^*\pi + u$
$\widehat{\pi}_{gls} = (X^{*\prime}(\Sigma \otimes I)^{-1}X^*)^{-1}X^{*\prime}(\Sigma \otimes I)^{-1}y$
$= ((I \otimes X')(\Sigma^{-1} \otimes I)(I \otimes X))^{-1}(I \otimes X')(\Sigma^{-1} \otimes I)y$ by property 2) and 3)
$= ((\Sigma^{-1} \otimes X')(I \otimes X))^{-1}(\Sigma^{-1} \otimes X')y$ by property 1)
$= (\Sigma^{-1} \otimes X'X)^{-1}(\Sigma^{-1} \otimes X')y$ by property 1)
$= (\Sigma \otimes (X'X)^{-1})(\Sigma^{-1} \otimes X')y$ by property 3)
$= (I \otimes (X'X)^{-1}X')y$ by property 1)

This is just OLS: $\widehat{\pi}_{gls} = (I \otimes (X'X)^{-1}X')y = \widehat{\pi}_{ols} = (X^{*\prime}X^*)^{-1}X^{*\prime})y$

2) Show V$(\widehat{\pi}|X) = \sum \otimes (X'X)^{-1}$

V$(\widehat{\pi}|X) = \Delta V(y|X)\Delta'$

where $\Delta = (I \otimes (X'X)^{-1}X')$
$V(y|X) = V((I \otimes X)\pi + u|X) = V(u|X) = \sum \otimes I$

V$(\widehat{\pi}|X) = \Delta V(y|X)\Delta' = (I \otimes (X'X)^{-1}X')(\sum \otimes I)(I \otimes X(X'X)^{-1})$ by property 2)
$= (\sum \otimes (X'X)^{-1}X')(I \otimes X(X'X)^{-1})$ by property 1)
$= \sum \otimes ((X'X)^{-1}X'X(X'X)^{-1})$ by property 1)
$= \sum \otimes (X'X)^{-1}$

3) Show $\widehat{\theta}_{gls} = (G'A^{-1}G)^{-1}G'A^{-1}\widehat{\pi}_{gls}$ where $A = V(\widehat{\pi}_{gls}|X)$

$y = (I \otimes X)G\theta + u$

$$\begin{aligned}\widehat{\theta}_{gls} &= (G'(I \otimes X)'(\Sigma \otimes I)^{-1}(I \otimes X)G)^{-1}G'(I \otimes X)'(\Sigma \otimes I)^{-1}y \\ &= (G'(\Sigma^{-1} \otimes X)'(I \otimes X)G)^{-1}G'(\Sigma^{-1} \otimes X)'y \\ &= (G'(\Sigma^{-1} \otimes X'X)'G)^{-1}G'(\Sigma^{-1} \otimes X)'y\end{aligned}$$

From above: $A = V(\widehat{\pi}_{gls}|X) = \sum \otimes (X'X)^{-1}$
$\widehat{\theta}_{gls} = (G'A^{-1}G)^{-1}G'(\Sigma^{-1} \otimes X)'y$

From above: $\widehat{\pi}_{gls} = (I \otimes (X'X)^{-1}X')y$
$\widehat{\theta}_{gls} = (G'A^{-1}G)^{-1}G'(\Sigma^{-1} \otimes X'X)(I \otimes (X'X)^{-1}X')y$
since $(\Sigma^{-1} \otimes X)' = (\Sigma^{-1} \otimes X'X)(I \otimes (X'X)^{-1}X')$

$\widehat{\theta}_{gls} = (G'A^{-1}G)^{-1}G'A^{-1}\widehat{\pi}_{gls}$

Question 2:

$y_{1i} = \alpha_1 + \beta x_i + \epsilon_{1i}$
$y_{2i} = \alpha_2 + \epsilon_{2i}$
Prove:
1) $\widehat{\alpha}_{1,gls} = \widehat{\alpha}_{1,ols}$ and $\widehat{\alpha}_{2,gls} = \widehat{\alpha}_{2,ols}$
where $\widehat{\alpha}_{1,ols} = \overline{y}_1 - \widehat{\beta}_{1,ols}\overline{x} = \overline{y}_1$ since $\overline{x}$ assumed zero.

and $\widehat{\alpha}_{2,ols} = \overline{y}_2$

2) $\widehat{\beta}_{gls} = \delta_1\widehat{\beta}_{1,ols} + \delta_2\widehat{\beta}_{2,ols}$
where $\widehat{\beta}_{1,ols} = \dfrac{\sum y_{1i}x_i}{\sum x_i^2}$ and $\widehat{\beta}_{2,ols} = \dfrac{\sum y_{2i}x_i}{\sum x_i^2}$

This is the two equation case from lecture notes.
Let $x_1 = [s\ x]$ and $x_2 = s$ where s is a vector of ones.

$b_{1,gls} = [\widehat{\alpha}_{1,gls}\ \widehat{\beta}_{gls}]$ and $b_{2,gls} = [\widehat{\alpha}_{2,gls}]$
$b_{1,ols} = [\widehat{\alpha}_{1,ols}\ \widehat{\beta}_{ols}]$ and $b_{2,ols} = [\widehat{\alpha}_{2,ols}]$

Proof of 1)

$$b_{2,gls} = b_{2,ols} - \frac{\sigma_{21}}{\sigma_{11}}(x_2'x_2)^{-1}x_2'(y_1 - x_1 b_{1,gls})$$
$$\widehat{\alpha}_{2,gls} = \widehat{\alpha}_{2,ols} - \frac{\sigma_{21}}{\sigma_{11}}(s's)^{-1}s'(y_1 - s\widehat{\alpha}_{1,gls})$$
i) $\widehat{\alpha}_{2,gls} = \overline{y}_2 - \frac{\sigma_{21}}{\sigma_{11}}[\overline{y}_1 - \widehat{\alpha}_{1,gls}]$

since $s's = N$, $(s's)^{-1} = 1/N$, and $s'y_1 = \sum y_{1i}$

By same argument,
$$b_{1,gls} = b_{1,ols} - \frac{\sigma_{12}}{\sigma_{22}}(x_1'x_1)^{-1}x_1'(y_2 - x_2 b_{2,gls})$$

ii) $\widehat{\alpha}_{1,gls} = \overline{y}_1 - \frac{\sigma_{12}}{\sigma_{22}}[\overline{y}_2 - \widehat{\alpha}_{2,gls}]$

Note:
$$(x_1'x_1) = \begin{bmatrix} N & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} = \begin{bmatrix} N & 0 \\ 0 & \sum x_i^2 \end{bmatrix}$$
$$(x_1'x_1)^{-1} = \begin{bmatrix} 1/N & 0 \\ 0 & 1/\sum x_i^2 \end{bmatrix}$$

Substitute ii) into i):
$$\widehat{\alpha}_{2,gls} = \overline{y}_2 - \frac{\sigma_{21}}{\sigma_{11}}[\overline{y}_1 - \{\overline{y}_1 - \frac{\sigma_{12}}{\sigma_{22}}[\overline{y}_2 - \widehat{\alpha}_{2,gls}]\}]$$
$$= \overline{y}_2 - \frac{\sigma_{21}}{\sigma_{11}}[\frac{\sigma_{12}}{\sigma_{22}}[\overline{y}_2 - \widehat{\alpha}_{2,gls}]]$$

$$= \overline{y}_2 - \frac{\sigma_{21}}{\sigma_{11}}\frac{\sigma_{12}}{\sigma_{22}}\overline{y}_2 + \frac{\sigma_{21}}{\sigma_{11}}\frac{\sigma_{12}}{\sigma_{22}}\widehat{\alpha}_{2,gls}$$
$$\widehat{\alpha}_{2,gls}[1 - \frac{\sigma_{21}}{\sigma_{11}}\frac{\sigma_{12}}{\sigma_{22}}] = [1 - \frac{\sigma_{21}}{\sigma_{11}}\frac{\sigma_{12}}{\sigma_{22}}]\overline{y}_2$$

Thus, $\widehat{\alpha}_{2,gls} = \overline{y}_2 = \widehat{\alpha}_{2,ols}$

Substitute again
$$\widehat{\alpha}_{1,gls} = \overline{y}_1 - \frac{\sigma_{12}}{\sigma_{22}}[\overline{y}_2 - \overline{y}_2]$$

Thus, $\widehat{\alpha}_{1,gls} = \overline{y}_1 = \widehat{\alpha}_{1,ols}$

Proof of 2)
$b_{1,gls} = b_{1,ols} - \frac{\sigma_{12}}{\sigma_{22}}(x_1'x_1)^{-1}x_1'(y_2 - x_2 b_{2,gls})$

$$\widehat{\beta}_{gls} = \widehat{\beta}_{1,ols} - \frac{\sigma_{12}}{\sigma_{22}}[\frac{x'y}{\sum x_i^2} - \frac{x's}{\sum x_i^2}\widehat{\beta}_{gls}]$$

Note: $(x_1'x_1)^{-1}x_1' = \begin{bmatrix} 1/N & 0 \\ 0 & 1/\sum x_i^2 \end{bmatrix}\begin{bmatrix} s' \\ x' \end{bmatrix}$

$$\widehat{\beta}_{gls} = \widehat{\beta}_{1,ols} - \frac{\sigma_{12}}{\sigma_{22}}[\widehat{\beta}_{2,ols} - \frac{\sum x_i}{\sum x_i^2}\widehat{\beta}_{gls}]$$

$$= \widehat{\beta}_{1,ols} - \frac{\sigma_{12}}{\sigma_{22}}\widehat{\beta}_{2,ols}$$

Thus, $\widehat{\beta}_{gls} = \delta_1\widehat{\beta}_{1,ols} + \delta_2\widehat{\beta}_{2,ols}$
where $\delta_1 = 1$ and $\delta_2 = -\frac{\sigma_{12}}{\sigma_{22}}$

Question 3:

$$y_1 = \gamma_1 y_2 + \beta_{11}x_1 + \beta_{21}x_2 + \beta_{31}x_3 + \varepsilon_1$$
$$y_2 = \gamma_2 y_1 + \beta_{12}x_1 + \beta_{22}x_2 + \beta_{32}x_3 + \varepsilon_2$$

a) Rewrite equations:
$$y_1 - \gamma_1 y_2 = \beta_{11}x_1 + \beta_{21}x_2 + \beta_{31}x_3 + \varepsilon_1$$
$$-\gamma_2 y_1 + y_2 = \beta_{12}x_1 + \beta_{22}x_2 + \beta_{32}x_3 + \varepsilon_2$$

$$\Gamma = \begin{bmatrix} 1 & -\gamma_2 \\ -\gamma_1 & 1 \end{bmatrix} \quad I - \Gamma = \begin{bmatrix} 0 & \gamma_2 \\ \gamma_1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & \gamma_2 \\ \gamma_1 & 0 \\ \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \end{bmatrix}$$

Check order condtion:
Eq. 1: m=2, $R_1 = 1$, not id.
Eq. 2: m=2, $R_2 = 1$, not id.

b) Each of these restrictions changes the A matrix

1) Eq. 1: m=2, $R_1 = 2$, just id.
Eq. 2: m=2, $R_2 = 2$, just id.

2) Eq. 1: m=2, $R_1 = 1$, not id.
Eq. 2: m=2, $R_2 = 3$, over id.

3) Eq. 1: m=2, $R_1 = 2$, just id.
Eq. 2: m=2, $R_2 = 1$, not id.

4) Eq. 1: m=2, $R_1 = 1$, not id.
Eq. 2: m=2, $R_2 = 2$, just id.
(I think my answer in the previous "suggested answers" was not correct.)

5) Eq. 1: m=2, $R_1 = 2$, just id.
Eq. 2: m=2, $R_2 = 1$, not id.

6) Eq. 1: m=2, $R_1 = 2$, just id.
Eq. 2: m=2, $R_2 = 1$, not id.

7) Eq. 1: m=2, $R_1 = 1$, not id.
Eq. 2: m=2, $R_2 = 1$, not id.

8) Eq. 1: m=2, $R_1 = 3$, over id.
Eq. 2: m=2, $R_2 = 3$, over id.

9) Eq. 1: m=2, $R_1 = 4$, over id.
Eq. 2: m=2, $R_2 = 3$, over id.

Question 4:

$$A = \begin{bmatrix} 0 & -\gamma_{12} & 0 & 0 \\ -\gamma_{21} & 0 & -\gamma_{23} & -\gamma_{24} \\ 0 & -\gamma_{32} & 0 & -\gamma_{34} \\ -\gamma_{41} & -\gamma_{42} & 0 & 0 \\ 0 & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & 1 & 0 & \beta_{24} \\ \beta_{31} & \beta_{32} & \beta_{33} & 0 \\ 0 & 0 & \beta_{43} & \beta_{44} \\ 0 & \beta_{52} & 0 & 0 \end{bmatrix}$$

Check order conditon:
eq.1, m=4, $R_1 = 5$, over id.
eq.2, m=4, $R_1 = 2$, not id.
eq.3, m=4, $R_1 = 5$, over id.
eq.4, m=4, $R_1 = 4$, just id.

Cannot identify the parameters in the second equation. Can identify parameters in all other equations.